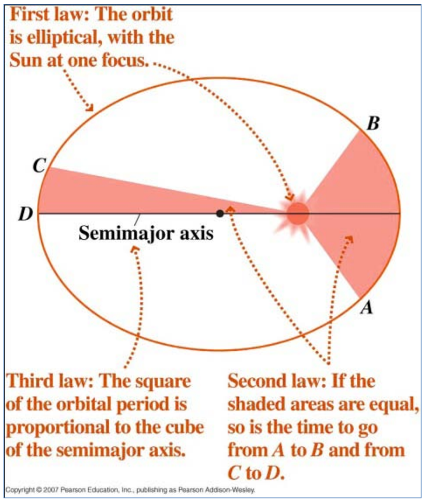
Lecture 17

Gravity and statics

Kepler's laws

Kepler's superb insights on the data taken on planetary motions made it possible for Newton to make his landmark contributions. Kepler's laws are summarized in this image. Regarding the first law, note that a circular orbit is a special case of an elliptical orbit. Regarding the second law, note that for a small displacement $d\vec{r}$ from the position $\vec{r}$, the area swept by the displacement is given by $dA = |\vec{r} \times d\vec{r}|/2$. [Why? The area swept by $d\vec{r}$ from $\vec{r}$ is the area of the triangle made by the two vectors $d\vec{r}$ and $\vec{r}$, which is half the area of the parallogram made by the same. The area of the parallogram is given by $|\vec{r} \times d\vec{r}|$.] Thus, $\frac{dA}{dt} = \frac{\left|\vec{r} \times \frac{d\vec{r}}{dt}\right|}{2} = |\vec{L}|/(2m)$, where $\vec{L}$ is the angular momentum ($m\vec{r} \times \vec{v}$). According to Kepler's 2nd law, $dA/dt =$ constant, and thus, $|\vec{L}|$ is constant. This and the fact the direction of $\vec{L}$ is perpendicular to the plane of motion (left as exercise for readers) means that $\vec{L}$ is conserved. In short, Kepler's 2nd law is, in modern language, the conservation of the angular momentum. The 3rd law is related to the nature of the force, and we will derive it in the special case of the circular orbit, below.



Newton's law of gravitation

According to this law, two masses (m_1 and m_2) attract each other and the magnitude of the attractive force is given by

$$F = \frac{Gm_1m_2}{r^2} \tag{17.1}$$

where r is the distance between the two masses, and G is the constant of universal gravitation $G = 6.67 \times 10^{-11}\ \mathrm{Nm^2/kg^2}$. This force is a so-called **inverse square law** force, since $F \propto r^{-2}$. [Later, it was discovered that the Coulomb force between electric charges also satisfies the inverse square law.]

Example 17.1 On the surface of the Earth, show that the gravitational force on a mass m is mg with $g = 9.81\ \mathrm{m/s^2}$, using Earth's mass and radius. At an altitude of 3.6×10^4 km ("geosynchronous orbit"), what is the value of "g"?

Solution. The gravitational force on a mass m at an altitide h is given by

$$F = \frac{GM_Em}{(R_E+h)^2} = mg_h \quad \text{with} \quad g_h = \frac{GM_E}{(R_E+h)^2} \tag{17.2}$$

g_h here is the generalization of g that we have been using throughout this course. For $h = 0$, $g_{h=0} = 6.67 \times 10^{-11} \times 5.97 \times 10^{24}/(6.37 \times 10^6)^2$ m/s^2, using $M_E = 5.97 \times 10^{24}$ kg and $R_E = 6.37 \times 10^6$ m. This value turns out to be $g_{h=0} = 9.81$ m/s^2. For other values of h, we can simply use, by virtue of the "inverse square law" nature

$$g_h = g_{h=0}\left(\frac{R_E}{R_E+h}\right)^2 \tag{17.3}$$

So, at $h = 3.6 \times 10^4$ km = 3.6×10^7 m = 5.65 R_E. And, thus, $g_h = 9.81 \times \left(\frac{1}{6.65}\right)^2 =$ 0.22 m/s^2.

Force field

Consider a small mass m pulled by a very large mass M. If $M \gg m$, then for all practical purposes M is stationary. Let us then take the origin of the coordinate system as the position of M. The gravitational force on m is then given by $\vec{F} = -\frac{GMm}{r^2}\hat{r}$, where $\hat{r}$ is the *radial unit vector*, a unit length vector whose direction is parallel to the position vector $\vec{r}$ of mass m. One often talks about m being in a **force field** $\vec{g}(r) = -\frac{GM}{r^2}\hat{r}$, and thus m experiencing the force $m\vec{g}(r)$. The small mass m is called a **test mass** in this context – it is a mass using which we can detect the force field by M without disturbing M.

Is this apparently very simple manipulation to define a mathematical object – force field – important? If so, why? The answer lies in the importance of "field" in modern physics. To physicists, and perhaps to all laymen too, it is a discomforting thought that two objects can influence each other when they are far from each other. However, if an object is generating a force field, which is then felt by another object, that is a totally different story. [This is the view of the "field theory" in physics, and all modern physics is the field theory in one form or another. From this point of view, "the force is everywhere" and the vacuum is filled with "force particles" that are responsible for force [plus a zoo of other particles]. Often times what modern physicists try to do is to shake the vacuum very hard, with a particle accelerator e.g., to figure out what "the vacuum is really made of."]

Newton's shell theorem (symmetry, symmetry, spherical symmetry ...)

Consider a mass distribution, which depends only on the distance r of the position vector $\vec{r}$. Such a distribution is said to have a "spherical symmetry." It means that the mass density does not depend on the direction at all, just like a sphere which looks the same regardless of the direction in which you look at it. An example. A simple model of the Earth would be that the mass density = constant (average mass density = 5.5 g/cm^3) for $r \le R_E$ and 0 otherwise. Or, it could be made more realistic by saying that the mass density ≈ 13 g/cm^3 at the core, decreasing to a surface value ≈2.2 g/cm^{3}as r approaches R_E. In either model, the mass density is spherically symmetric, as long as we ignore the variation of the mass density as a function of latitude and longitude.

Newton's shell theorem states that the gravitational force at a position $\vec{r}$ due to the presence of a spherically symmetric mass distribution is determined *only* by the mass distribution inside the radius $r = |\vec{r}|$. The mass distribution outside the radius r exerts no force, since their forces exactly cancel out! This is due to the inverse square law. [The same kind of law for Coulomb force, another inverse square law force, goes by the name of Gauss's law.] More specifically, the force a test-mass m at $\vec{r}$ experiences due to a spherical mass distribution is given by

$$\vec{F} = -G\frac{mM(r)}{r^2}\hat{r} \quad \text{where } M(r) = \text{the total mass } \underline{\text{inside}} \text{ radius } r \qquad (17.4)$$

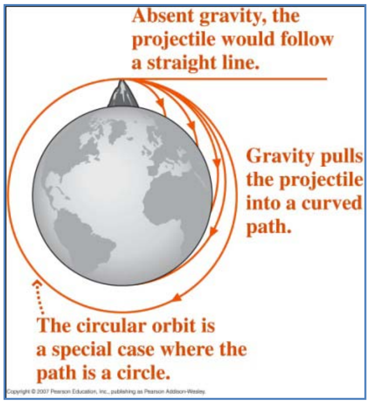
Example 17.2 Journey through the center of the Earth. Just for fun, let us assume that you can dig a hole through the Earth from North pole to South pole. You will also need Iron Man's suit, or something like it to withstand all the heat that you will encounter near the core. Anyhow, suppose then you drop into the hole. What kind of motion will you have? How long does it take for you to reach the other side of the Earth? Assume that the density of the Earth is uniform and there is no air resistance.

> *Solution*. Suppose you are at the radius r ($< R_E$) from the center of the Earth. By Newton's shell theorem, the gravity that you feel is determined by the mass of the Earth inside radius r, which is $M_E r^3/R_E^3$. So, the force that you feel is $\hat{F} = -\frac{Gm\left(\frac{M_E r^3}{R_E^3}\right)}{r^2}\hat{r} = -\frac{GmM_E}{R_E^3}r\hat{r}$. This is precisely a Hooke's law force for a one dimensional motion! So, the motion that you will experience is a simple harmonic motion! The "spring constant" in this case is $k = GmM_E/R_E^3$, and so the angular frequency $\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{GM_E}{R_E^3}}$. And thus the period $T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{R_E^3}{GM_E}}$. This is merely 84 minutes. To go from North pole to South pole, it takes only half of it!

Motion in gravitational field ("falling" in different styles)

Newton made a remarkable discovery by recognizing, courageously, that an apple falling from a tree is governed by the same law as the Moon going round the Earth. In physics, this type of principle is called the "symmetry principle" – namely the physical law should be valid here and there alike, in the past and in the future alike, in this direction and in that direction alike, and in this reference frame and in that reference frame alike. And so on and so forth. Einstein is the first person who had the courage and insight to put the symmetry principle in front of all other principles, and paved the way to modern physics as we know it.

So, how is it that the apple falling from a tree is governed by the same law as the Moon going around the Earth? This has to do with different initial conditions. If initially the object does not have any velocity, it will simply fall in the radial direction. That would be the apple. However, if the object has the right speed, moving tangentially, i.e. perpendicular to the radial direction, then it can do a circular motion. That would be the Moon. See the diagram on the right.



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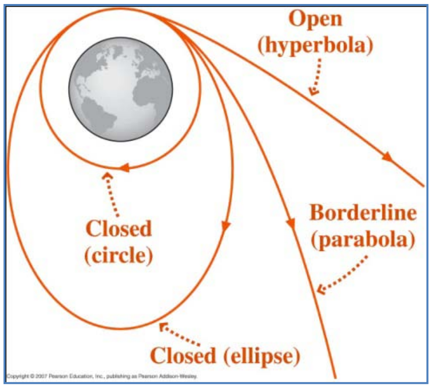
The circular orbit, it turns out, is only one possibility. The following diagram shows all possible shapes of orbitals, if the initial speed is not zero. [In this global view, what we called a "parabola" for a projectile motion turns out to be a small section of an ellipse.]



Example 17.3 A circular satellite orbit with the period T = 1 day is called a **geosynchronous orbit**. What is the altitude for the geosynchronous orbit of the Earth?

Solution. The key observation here is that the centripetal acceleration $mR\omega^2$ is provided by the gravitational force = $G\frac{M_E m}{R^2}$ where R is the orbit-radius and m is the mass of the satellite. Thus we have $mR\left(\frac{2\pi}{T}\right)^2 = G\frac{M_E m}{R^2}$, which means $GM_E T^2 = 4\pi^2 R^3$. **This is Kepler's 3rd law, proved for a circular motion**. Now, for altitude h, we should use $R = R_E + h$, and thus $h = \sqrt[3]{\frac{GM_E T^2}{4\pi^2}} - R_E$. For $T = 1$ day, we get $h = 3.6 \times 10^4$ km.

Gravitational potential energy

For the given force $\vec{F} = -\frac{GMm}{r^2}\hat{r}$, what is the potential energy funnction? The answer:

$$U(r) = -\frac{GMm}{r} \tag{17.5}$$

Why? Because F (radial direction) $= -\frac{dU}{dr} = GMm\frac{d(r^{-1})}{dr} = -\frac{GMm}{r^2}$. The existence of $U(r)$ means that the gravitational force is conservative.

The conservation of total mechanical energy in a gravitational field

$$E = K + U(r) = \frac{1}{2}mv^2 - \frac{GMm}{r} = \text{constant of motion} \tag{17.6}$$

It should be noted that here $v^2 = \vec{v}\cdot\vec{v}$, where $\vec{v}$ is in general a *three dimensional* vector. [Although for any orbits that we considered above, a two dimensional vector would suffice.]

Escape speed

What goes up may not come down! If the total mechanical energy is zero, then from the above equation, it can be seen that at $r = \infty$ the kinetic energy is 0. This is the minimum condition for an object to escape the gravitational field. For this reason, the escape speed, v_{esc}, at r is defined as the speed for which $E = 0$. Thus, $\frac{1}{2}mv_{esc}^2 - \frac{GMm}{r} = 0$, which means $v_{esc} = \sqrt{\frac{2GM}{r}}$. For the Earth gravity, the escape speed at the surface of the Earth = 11.2 km/s (i.e. 22 thousand mph).

Binding energy

Suppose that an object is orbiting in a gravitational field with energy $E < 0$. Then, $-E$ is its **binding energy**. That is, if the object gains that binding energy somehow, it will be able to escape the gravitational field! The states with energy $E < 0$ are called "**bound states**." [The notion of binding energy or bound states is more commonly used in atomic systems, whose quantum mechanical problem of charge in a Coulomb field quite resembles the current problem of mass in a gravitational field.]

Circular motion in a gravitational field

For a circular motion of mass m, in a gravitational field due to a much larger mass M, some simple useful relations can be derived. $GMT^2 = 4\pi^2 R^3$ (Kepler's third law) is one of them (proved in the Example 17.3). There are additional ones. Note that for a circular motion, the potential energy $U = -\frac{GMm}{R}$ is constant, since R is constant. Since the total mechanical energy $E = K + U$ is constant, it then follows that $K = \frac{1}{2}mv^2$ is constant. That is, a circular motion in a gravitational field is *necessarily* a uniform circular motion. Now, consider the centripetal force, which comes from the gravitational force: $\frac{mv^2}{R} = \frac{GMm}{R^2}$. Multiplying both sides by R, we get $mv^2 = \frac{GMm}{R}$. Namely, $2K = -U$. Therefore, for a circular motion in a gravitational field, we have

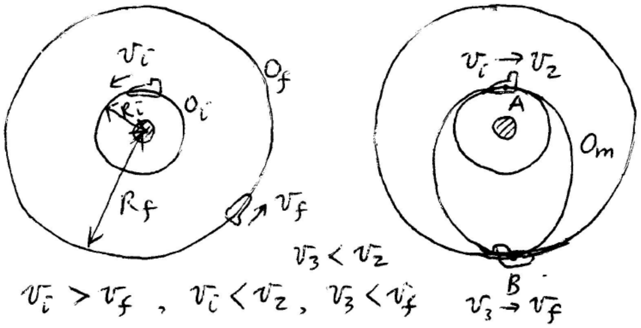
$$E = K + U = -K = \frac{1}{2}U < 0 \tag{17.7}$$

That is, any object doing a circular motion in a gravitational field is in a bound state. Namely, they can't escape the gravitational field, unless some energy is delivered to it from outside.

Example 17.4 Express the velocity of a mass m, that is doing a circular motion in a gravitational force field by a large mass M, as a function of R and other parameters.

Solution. We just showed that $mv^2 = \frac{GMm}{R}$, which means $v = \sqrt{\frac{GM}{R}}$. This may seem a bit paradoxical. Namely, as R increases, the speed decreases, while $E = -K = \frac{1}{2}U$ increases!

Example 17.5 Shuttle orbit change paradox. You are in charge of a space shuttle orbiting the Earth in a circle. You want to increase its energy so that it goes into a circular orbit with a greater radius. You fire the jet engine once (for a very short time), wait for a set time, and fire the engine again. Each time you fire the jet engine to give a forward thrust: the momentum and thus the kinetic energy of the satellite *increase* each time. And yet, the kinetic energy of the final circular orbit is actually lower than the kinetic energy of the initial circular orbit, as follows from the previous example. Solve this apparent paradox.



Solution. Suppose the initial circular orbit, O_i, has the radius R_i and the speed v_i and the final circular orbit, O_f, has the radius R_f and the speed v_f. From the previous example, $\frac{v_f}{v_i} = \sqrt{\frac{R_i}{R_f}}$, and so the kinetic energy must decrease, going from O_i to O_f. How can this happen when all you do is to increase the kinetic energy by forward thrusts? The answer lies in *how* you go from O_i to O_f. You have to go through an intermediate orbit, e.g. O_m in the diagram. You can do the following maneuvers. First, fire the jet engine quickly at point A of the orbit so that the orbit becomes O_m, an ellipse. The amount (f) and the time duration (Δt) of the forward thrust determines the new speed v_2, through $f\Delta t = \Delta p = mv_2 - mv_i > 0$ (forward thrust: both momentum and kinetic energy increase). Here, Δt should be short enough so that the shuttle is basically at point A while

the thrust is being applied. You apply the exact amount of impulse so that O_m is an orbit that is tangential to both O_i and O_f: the closest point (the perigee) occurs at point A, which touches O_i, and the farthest point (the apogee) occurs at point B, which touches O_f. Second, wait until the shuttle comes to point B, where your space shuttle's speed is now $v_3 < v_2$. This inequality, which you can understand from Kepler's 2nd law, is the most important piece for solving the puzzle here. Third, fire your jet engine at point B, so that the speed increases from v_3 to v_f. In these maneuvers, the difference $v_2 - v_3$ is greater than the combined increase in speed $v_2 - v_i$ and $v_f - v_3$, and so the space shuttle has no trouble satisfying $v_f < v_i$.
Quantitatively, here is how it goes. The exact values of v_2 and v_3 can be solved for, using $R_i v_2 = R_f v_3$ (Kepler's 2nd law) and the energy conservation equation (with m cancelled out) $\frac{1}{2}v_2^2 - \frac{GM}{R_i} = \frac{1}{2}v_3^2 - \frac{GM}{R_f}$. The result: $v_2 = \sqrt{\frac{2R_f}{R_i+R_f}}\,v_i$ and $v_3 = \sqrt{\frac{2R_i}{R_i+R_f}}\,v_f$.

Statics

Statics is related to the load and the stability of, typically large, objects such as buildings, bridges, statues, etc. One can recognize why the topic of statics is very important in practice. As we will see, the principle of statics is very simple. This, by the way, does not mean that problems in statics are simple; they are in general quite interesting, and you should try some problems in the back of Chapter 12 (like 39, 56, e.g., but only if you have time!).

Statics is a subject dealing with static equilibrium. What is static equilibrium? It means that everything is at rest (static) and everything is at equilibrium (no net force). From this definition, note that as long as a body is in static equilibrium we can consider it a rigid body. Therefore, for a body in static equilibrium, the following conditions must necessarily hold.

$$\text{Net external force = zero,}\quad \text{Net external torque } \textit{around any point} \text{ = zero.} \tag{17.8}$$

The first condition ensures that the center of mass of the body, once at rest, will remain at rest, according to Eq. (16.3). For the second condition, recall that the only internal motion that a rigid body can have is rotation around the center of mass (Eq. (14.10); also see Eq. (14.4)). So, a rigid body in static equilibrium must have no net external torque around the center of mass. However, this condition can be made much more general, as stated above. Namely, it turns out that in static equilibrium, the net external torque is zero around *any* point. Physically, this means that there is no rotation around any point. This is because, if $\sum_i \vec{F}_i = 0$, where the sum is over all external forces $\vec{F}_i$, then we can prove the following: if the net external torque is zero around *one* point (like, but not necessarily, the center of mass) then the net external torque is zero around *any* point. [Proof: Call the first point O, and the second (arbitrary) point P. Define the position vector of $\vec{F}_i$ as $\vec{r}_{i,O}$ when referenced to point O, and $\vec{r}_{i,P}$ when referenced to point P. Then, $\vec{r}_{i,O} = \vec{r}_{i,P} + \vec{R}_{P,O}$, where $\vec{R}_{P,O}$ is the vector from P to O. So, if $\sum_i \vec{r}_{i,O} \times \vec{F}_i = 0$, then we get
$0 = \sum_i(\vec{r}_{i,P} + \vec{R}_{P,O}) \times \vec{F}_i = \sum_i \vec{r}_{i,P} \times \vec{F}_i + \sum_i \vec{R}_{P,O} \times \vec{F}_i = \sum_i \vec{r}_{i,P} \times \vec{F}_i + \vec{R}_{P,O} \times \sum_i \vec{F}_i = \sum_i \vec{r}_{i,P} \times \vec{F}_i$, where in the last step $\sum_i \vec{F}_i = 0$ is used. QED.]

This last point is of great importance in doing problems of statics. To set up the equation for the total external torque, one can pick *any* point around which to calculate the torque. If one establishes that the torque is zero around that point, then one has proven that the torque is zero around any other point, as long as the net eternal force is also zero. Then it should become obvious that **the choice of the point around which the torque is calculated should be made so that the calculation is the least complicated!**

One note about **center of gravity** and **center of mass**. As is obvious from the above discussion and, more fundamentally, the definition of the torque ($\vec{\tau} = \vec{r} \times \vec{F}$), it is important to know where the external force is applied, since the position at which the force is applied is crucial for the calculation of the torque. This leads to the concept of the **center of force**, such as the **center of gravity**, the position at which the net gravity is applied. In general, the center of gravity is different from the center of mass. For non-uniform gravitational field, such as the Moon experiences of the Earth, the center of gravity of the Moon is closer to the Earth than the center of mass of the Moon, due to the inverse square law nature of the force. However, for a uniform gravitational field, approximately valid near the surface of the Earth, the center of gravity is identical with the center of mass.

Example 17.6 An object is resting on a table. Suppose that the contact area, A, between the object and the table consist of only one simple totally convex shape: this means the normal force by the table on the object is applied at some point, Q, which is within, or on the boundary of, A. Q = center of normal force. Consider point P, the intersection point between the table and the vertical line dropped down from the center of mass of the object. Show that for the object to be in static equilibrium, P should be contained within, or on the boundary of, A. Also, show that Q = P.

> *Solution*. Suppose P lies outside A. Then Q and P are distinct. Choose any point between Q and P, like the midpoint, and call it R. First notice that for this object to be in static equilibrium, the normal force and the gravity should sum up to zero force. This leads to the familiar result: the normal force should point up while the gravity points down. However, this means that, around point R, the torques due to the normal force and the gravity are in the same direction! They both tend to topple the object. So, if P is outside A, then there cannot be any static equilibrium. Conversely, for the object to be in static equilibrium, P should lie within, or on the boundary of, A. Furthermore, it follows that the center of normal force, Q, must coincide with P, because only then the total external torque can be zero.

Got it? 12.1-3 and Example 12.2 are essential to understand.

– FIN – Viva compassion, viva curiosity!