Lecture 16

Many particles, conservation of momentum and collision

In physics we often speak about a “**classical object**.” This is an object that satisfies Newton’s 2nd law, $\vec{F} = d\vec{p}/dt$. So, all objects that we consider in this course are classical objects. [Note that atoms in hydrogen molecules or NaCl molecules are not classical objects. In some examples/problems, we just pretended that they were. That pretension turns out to be OK as long as the simple description of the stable equilibrium or the bond length goes, but is ultimately unforgivable in view of other quantum mechanical properties of atoms and molecules.] These are big objects – roughly speaking big enough for us to see – that move much more slowly in comparison to light. Once we have a classical object, a collection of a few classical objects also satisfies Newton’s law. For example, we saw that the angular momentum was conserved in a system of objects, which consist of a merry-go-round, a boy and a girl. Not only this, each one classical object can be divided into many parts (like in calculus) each of which is another classical object. In short, Newton’s law is inherently a law about **many particles**, whether “particle” means separate classical objects forming one compound object or parts of a single object.

Center of mass and its importance

For a many particle system, the **center of mass** is an important concept. We defined it before, but here it is, one more time.

$$\vec{R}_{cm} = \frac{\sum_i \vec{r}_i m_i}{\sum_i m_i} = \frac{\sum_i \vec{r}_i m_i}{M}\,(\text{discrete}), \quad \frac{\int \vec{r}\,dm}{\int dm} = \frac{\int \vec{r}\,dm}{M}\ (\text{continuous}) \qquad (16.1)$$

Here, M is the total mass. Since the continuous case can be considered as an essentially discrete case also, throughout this lecture note, we will use the first form. However, mathematically, it is just as easy to use integrals instead of sums in all equations of this lecture note.

Why is the center of mass important? First, consider the total momentum of the system, $\vec{P} = \sum_i m_i \vec{v}_i = \sum_i m_i \frac{d\vec{r}_i}{dt}$. Because $\sum_i \vec{r}_i m_i = M\vec{R}_{cm}$, by taking the time derivative of both sides, we would get $\vec{P}$!

$$\vec{P} = \sum_i m_i \vec{v}_i = M\vec{V}_{cm} \ \text{ where } \ \vec{V}_{cm} = d\vec{R}_{cm}/dt \qquad (16.2)$$

So, the total momentum is just the total mass times the velocity of the center of mass.

Let’s take a second to really appreciate this fact. The system (or a compound object) that we are considering here may be “a space shuttle with astronauts moving about doing their jobs in and around it,” “a merry-go-round with several kids jumping up/down/in/out-of it,” or “the solar system consisting of the Sun plus planets, moons and asteroids.” No matter what may be happening between different parts of a compound object, though, the total momentum is simply given by $M\vec{V}_{cm}$!

Here is another equation of importance.

$$\vec{F} = \frac{d\vec{P}}{dt} \tag{16.3}$$

Here, $\vec{F}$ is the **net external force**. This equation is not necessarily a new equation, but now may feel more informative since we have now defined the notion of the center of mass explicitly. In this course, we've been busy describing motions of baseball, skateboarder, and spit of archer fish, and so on and so forth. What we've been doing was actually describing the motion of a center of mass, just as described by Eq. (16.3). As before, $\vec{F}$ is the net force. More precisely speaking, $\vec{F}$ is the **net external force**. Recall that net means total. Why only external? Because all internal forces are paired and they cancel out by Newton's 3rd law.

Again, let us take a pause, and appreciate the meaning of Eq. (16.3). Take the example of a space shuttle, with astronauts doing all kinds of things in and around it. The space shuttle may be doing a spin around its center of mass. According to Eq. (16.3), whatever these motions may be, the center of mass moves like a point particle with the total mass (shutttle + astronauts) under the influence of external forces only! Thus, as far as the motion of the center of mass ("average motion") is concerned, it is *utterly* unimportant what the astronauts are doing or whether or not the shuttle itself is spinning around its center of mass. The only thing that is important for the "average motion" is the external force. Another example is a ski jumper, as shown in the image. The jumper may be moving limbs and skis in complicated ways. No matter. The center of mass follows a trajectory that is determined only by external forces (gravity and air resistance).

Finally, the following equation can be derived for the total kinetic energy $K = \frac{1}{2}\sum_i m_i v_i^2$.

$$K = \frac{1}{2} M V_{cm}^2 + K_{int}, \quad K_{int} = \frac{1}{2}\sum_i m_i v_{i,cm}^2 \tag{16.4}$$

Here, $V_{cm} = |\vec{V}_{cm}|$, where $\vec{V}_{cm}$ is the velocity of the center of mass and $v_{i,cm} = |\vec{v}_{i,cm}|$, where $\vec{v}_{i,cm} = \vec{v}_i - \vec{V}_{cm}$. Namely, $\vec{v}_{i,cm}$ is the velocity of m_i in the center of mass reference frame, i.e. the frame in which the center of mass is at rest. So, the total kinetic energy is nicely separated into two terms – the kinetic energy of a point mass M moving with $\vec{V}_{cm}$, and the internal kinetic energy term, which accounts for individual mass's motion with respect to the center of mass. [Note: a similar decomposition is possible also for the total angular momentum.] In fact, we already discussed an example like this: for a rigid body, K_{int} was the rotational kinetic energy $\frac{1}{2} I_{cm}\omega^2$. The above equation is a general one, applicable to any system of particles, not just to a rigid body.

Principle of momentum conservation

If the net external force is zero, then the total momentum is constant.

$$\vec{P} = \sum_i m_i \vec{v}_i = M\vec{V}_{cm} = \text{constant (conserved) if net external force} = 0. \qquad (16.5)$$

This is a direct consequence of Eq. (16.3). We already stated this principle at the end of LN 14, where the two conservation laws (angular momentum conservation and linear momentum conservation) were stated together. Indeed, those two laws are both very fundamental, and it is good to remember them together.

Collision

Despite the name, collision is a very much cherished concept in physics. Why? Because the word "collision" not only refers to catastrophic every day collisions but also to controlled collisions between fundamental particles in physics experiments, which give us very important clues about Nature's laws.

Here, we consider collisions between everyday objects, for which the interaction during the collision is a short-ranged ("contact") electro-magnetic interaction. For such collisions, it is a very good approximation to assume that the objects no longer interact when they are well separated from each other.

We define **collision** as a very brief interaction between objects. Before and after a collision there is no interaction – i.e. it is as though those objects do not know each other.

Collisions are usually very dramatic processes. What happens during the collision? A severe deformation through exchange of forces. Some high speed photos are shown here to illustrate the point. Notice how flat the golf ball or the tennis ball becomes at the contact, where there is a momentary exchange of a large amount of forces.

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To be concrete, let us consider a collision between two objects, A and B. Let us say Δt is the duration of the collision. By our definition, this is a very short time (about 0.5 milli-seconds for that golf ball). During Δt, let us say that object A experiences an average force $\vec{F}_{ave}$. Then, by Newton's 3rd law, object B experiences $-\vec{F}_{ave}$. The **impulse** on object A is defined as

$$\vec{J}_A = \vec{F}_{ave}\,\Delta t \qquad \text{(impulse)} \qquad (16.6)$$

Note that during a collision, the force of collision is the most dominant one, and so from Newton's 2nd law, $\vec{F}_{ave} = \Delta\vec{p}/\Delta t$. Thus,

$$\vec{J}_A = \Delta\vec{p}_A \qquad \text{(impulse causes change of momentum)} \qquad (16.7)$$

This is easy to understand. A baseball hit by a bat flies off, with a great change of momentum. [Another example where the impulse is important is the "pulling the table cloth trick." You may have seen a restaurant worker who pulls the table cloth so fast that cups and plates on top of it remain virtually undisturbed while the table cloth is pulled away! By pulling the table cloth very quickly, Δt is made very small, and thus a very small change of momentum results on those objects that exchange the friction force with the table cloth.] Notice that, by

Newton's 3rd law, the impulse on object B will be $\vec{J}_B = \Delta\vec{p}_B = -\Delta\vec{p}_A$, and so the total momentum $\vec{P} = \vec{p}_A + \vec{p}_B$ is conserved, since $\Delta\vec{P} = \Delta\vec{p}_A + \Delta\vec{p}_B = 0$. This is nothing new. It is a mere re-verification of Eq. (16.5), applied to the motion of a compound object, consisting of objects A and B, *during the short time interval* Δt. [For a similar reason, the angular momentum is also conserved during a collision.]

So, we come to a very important fact about collisions.

The total momentum is conserved during any collision. (16.8)

One important word here is "during." If two objects collide and then they move up and down under the influence of gravity, then the total momentum will no longer be conserved in the *post*-collision motion, since the net external force is not zero. During a collision, however, the impulse by those external forces such as gravity is negligible in comparison to the impulse of the contact forces.

Elastic collision

In some collisions, the **total kinetic energy** of objects is conserved. If this is the case, then the collision is called an **elastic collision**.

Inelastic collision

A collision during which the total kinetic energy is not conserved is called an **inelastic collision**. If [two] objects collide and become one as the result of collision, then the collision is called a **totally inelastic collision**.

These distinctions can be explained as follows. Note that during a collision, the two (or however many) objects in a collision are severely deformed. What this means is that during the collision, their *internal* potential energies change. [These potential energies are electromagnetic potential energies. One way to visualize them is by viewing each molecule in an object as a little spring, like in Example 7.7 of text. All of those springs at the contact get squished during a collision. Since an object is likely in a stable equilibrium before collision, its potential energy likely goes up during the collision. However, the object might obtain a lower internal potential energy value by deformation also. Although it is not a "collision" problem per se, the figure skater drawing limbs in or the pulsar core becoming a neutron star examples discussed in a previous lecture is such an example where the deformation leads to a lower internal potential energy.] If the objects are completely elastic, then it means that, when the collision is over, they go back to the original state, as though nothing happened. The internal potential energy goes back to the original value, the total mechanical energy is conserved, and thus the total kinetic energy is conserved. If this sounds too good to be true, then you are correct. In real life, an elastic collision exists only in an approximate sense. Why? During a collision, an object experiences a permanent "damage" (a slight deformation). Also, the mechanical energy can be converted to thermal energy (heat due to friction) or sound energy. The extreme opposite case to the elastic collision is the totally inelastic collision. In the latter case, objects get stuck together and become one as the result of the collision, and therefore their internal potential energy changes occurring during the collision remain permanent. [If the two masses do not change during a collision, then one can show that the total kinetic energy always decreases in a totally inelastic collision. Note, however, that, in some collisions, masses do change because of the famous equation of Einstein, $E = mc^2$. This "mass-changing" physics or "what is mass" physics, while extremely interesting, is beyond the scope of this course.]

Example 16.1 Elastic collision in one dimension. Suppose there is a mass M at rest. A mass m slams on mass M head-on with a velocity v_0. What is the velocity v of mass m and V of mass M after the collision, which we assume to be an elastic collision?

Solution. The total momentum in the initial state: mv_0. The total momentum in the final state: $mv + MV$. OK, at this point, let us make sure that we know our coordinate system. Define the x axis to point along the initial velocity of mass m. So, v_0 is a positive number, by definition. We do not know what the sign is for v and for V (we expect the sign of V to be positive, though!). Our solution will let us know the sign of v and V, and thus in which direction m and M are moving in the final state. By momentum conservation, we have $mv_0 = mv + MV$. Since the collision is assumed to be elastic, we have $\frac{1}{2}mv_0^2 = \frac{1}{2}mv^2 + \frac{1}{2}MV^2$, which means $mv_0^2 = mv^2 + MV^2$. Note that this problem is solvable. Why? We have two unknowns, v and V, and two equations. So, solvable. Eliminate v from the momentum equation. $mv_0^2 = (mv_0 - MV)^2/m + MV^2$. $M(M+m)V^2 - 2mv_0MV = 0$. So, $V = 0$ or $\frac{2m}{M+m}v_0$. $V = 0$ corresponds to no collision. So keep only $V = \frac{2m}{M+m}v_0$. Then, $v = v_0 - \frac{MV}{m} = \frac{m-M}{m+M}v_0$. To summarize:

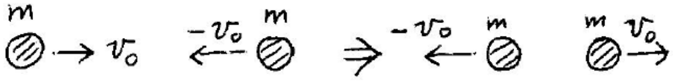
$$m \text{ moves at } v_0,\ M \text{ at rest} \rightarrow m \text{ at } v = \frac{m-M}{m+M}v_0,\ M \text{ at } V = \frac{2m}{M+m}v_0 \tag{16.9}$$

Considering the velocity of the mass m relative to mass M:

$$v - V = \frac{m-M-2m}{M+m}v_0 = -v_0 \tag{16.10}$$

This is the general property of a 1D elastic collision – the relative velocity of one mass to the other simply reverses the sign due to the collision! [The corresponding property in *any* – two is general enough, in effect – dimensions is that the *magnitude* of the relative velocity is conserved for an elastic collision between two objects.] It is instructive to examine three regimes of the full solution (16.9). (1) If $M \gg m$, then $v = -v_0$ and $V = 0$. This is, e.g., an elastic ball bouncing off the Earth. The super-massive Earth doesn't budge, while the elastic ball will simply reverse the direction (that is what the negative sign in $v = -v_0$ means), while keeping its speed. Assuming no air resistance, then, an elastic ball dropped from a certain height will bounce back to the same height. In real life, it is impossible to find such a ball, since the real collision is never truly elastic. (2) If $M = m$, then $v = 0$ and $V = v_0$. In this case, the incident mass stops and the target mass moves with the same velocity as the initial velocity of the incident mass. A real life example would be a billiard ball (with no spin or "English") hitting another billiard ball head-on. (3) If $M \ll m$, then $v = v_0$ and $V = 2v_0$. This is simply the same situation as (1), but with the roles of M and m reversed. So, let us express all velocities in the reference frame of m in the initial state (before collision). Mathematically, this means subtracting v_0 from all velocities. In this frame, m is at rest, M moves at $-v_0$ → m is at rest and, M moves at v_0.

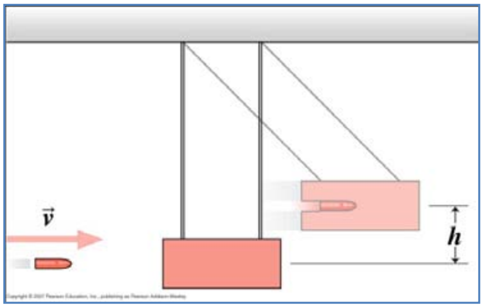
Example 16.2 Two identical masses do a head-on elastic collision with the velocities v_0 and $-v_0$. What are the velocities in the final state?



Solution. **Example 16.1 provided**

an effectively *complete and general* solution for a 1D elastic collision. How come? Wasn't the initial condition a special one (M at rest) rather than a general one? *Well, despite this, the fact that any reference frame which is not accelerating relative to an inertial frame is another inertial reference frame makes the solution of Example 16.1 an effectively complete solution.* Here is what we mean: if mass M *is* moving initially, then re-frame the question using the rest frame of mass M, use the result of the previous example, and then covert the velocity values back to those of the original reference frame. Here it goes for this particular example. First, use the reference frame in which the second mass is at rest. This means subtracting $-v_0$, i.e. adding v_0. In this new reference frame, the first mass is moving at $2v_0$, while the second mass is, of course, at rest. Second, by using the result of the previous Example (limit (2)), we know then that, in the final state, the first mass is at rest and the second mass moves at $2v_0$. Third, come back to the original reference frame, by subtracting v_0. So, answer = $-v_0$ and v_0, i.e. each velocity is simply reversed.

Example 16.3 Totally inelastic collision (Ex. 9.10 of textbook). **Ballistic pendulum**. If block+bullet goes up by h, what was v? Block mass = M, bullet mass = m. Initial velocity of bullet = v.



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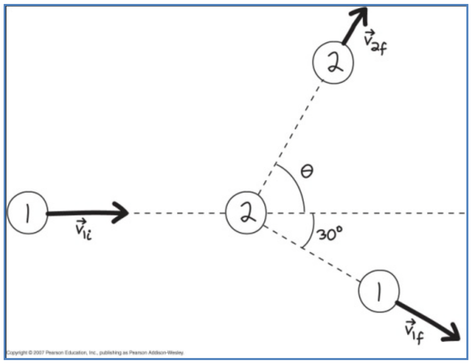
Solution. This is the case of a total inelastic collision, since the two (block and bullet) become one in the final state. Also, one should keep in mind that the initial state and the final state of the collision both occur during a short time interval when the bullet hits the block, while the height remains practically unchanged. So, for the collision, the momentum conservation equation $mv = (m+M)v_f$ is the only equation to use. After the collision, the pendulum swings up, and the total energy is conserved. $(M+m)gh = \frac{1}{2}(M+m)v_f^2$, i.e. $v_f = \sqrt{2gh}$. Plugging this into the momentum conservation equation, we get $v = \frac{M+m}{m}\sqrt{2gh}$. A way to measure the speed of the bullet.

Example 16.4 Elastic collision in two dimensions (Ex. 9.12 of textbook). A ball strikes a stationary ball of the same mass, and comes out at 30 degrees. In what direction is the other ball moving?



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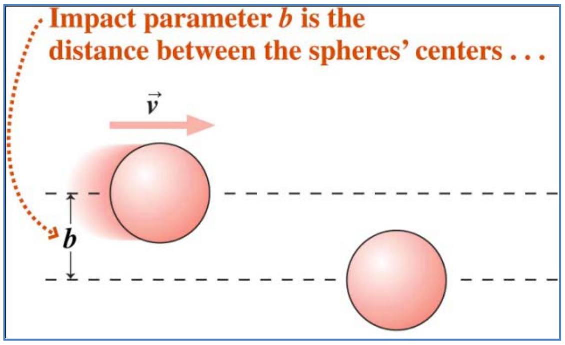
Solution. The answer is 60 degrees, since it is well-known that, for a 2D collision of identical masses with one mass at rest, the two masses move at 90 degrees with respect to each other in the final state. A good pool player knows and makes use of this fact! Let us see how this can be derived mathematically. First, the momentum conservation $m\vec{v}_{1i} = m\vec{v}_{1f} + m\vec{v}_{2f}$. Second, the energy conservation $\frac{1}{2}mv_{1i}^2 = \frac{1}{2}mv_{1f}^2 + \frac{1}{2}mv_{2f}^2$. By virtue of all masses being equal, these equations become simple: (1) $\vec{v}_{1i} = \vec{v}_{1f} + \vec{v}_{2f}$ (1) and (2) $v_{1i}^2 = v_{1f}^2 + v_{2f}^2$. (1) means $\vec{v}_{1i}\cdot\vec{v}_{1i} = \left(\vec{v}_{1f} + \vec{v}_{2f}\right)\cdot\left(\vec{v}_{1f} + \vec{v}_{2f}\right)$, which leads to $v_{1i}^2 = v_{1f}^2 + v_{2f}^2 + 2\vec{v}_{1f}\,\cdot$

$\vec{v}_{2f}$. Comparing this with (2), we conclude that $2\vec{v}_{1f} \cdot \vec{v}_{2f} = 0$, i.e. $\vec{v}_{1f}$ and $\vec{v}_{2f}$ are perpendicular/orthogonal to each other.

Elastic collision in two dimensions



An elastic collision between two particles in two dimensions is not as easy to solve as an elastic collision in one dimension (Example 16.1). The reason is that in the final state, there are four unknowns (two velocities, each velocity having two components), while the momentum conservation (a vector equation – so there are two equations) and the energy conservation (a scalar equation – so this is one equation) give only three equations. Four unknowns require four equations, not three equations, for a complete solution. What is the missing information? **The angular momentum**! Consider a ball coming and hitting another ball at rest in the above diagram. The vertical distance between the two balls is called the **impact parameter**, b. Notice that the angular momentum of the incident ball relative to the ball at rest is given by $L = mvb$ [the easiest way to see this is to break down – "resolve" – the position vector of the incident ball to the component parallel to $\vec{v}$ and the component perpendicular to $\vec{v}$; only the 2nd component gives rise to L] which is **the total angular momentum conserved in this collision**. With the addition of impact parameter, the elastic collision in two dimensions becomes completely solvable. For instance, the impact parameter is what determines what angle the incident ball will be deflected – 30 degrees in the last example.