Lecture 14

Rotational Motion (continued)

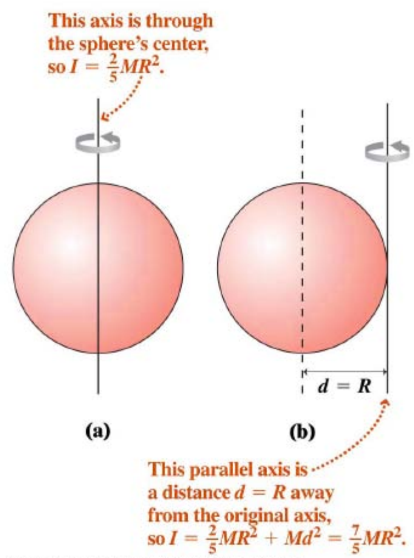
Center of mass

The center of mass of an object is defined as

$$\vec{R}_{cm} = \frac{\sum_i \vec{r}_i m_i}{\sum_i m_i}, \quad \frac{\int \vec{r}\, dm}{\int dm} \qquad (14.1)$$

In words, this definition means that the position of the center of mass of an object is defined as the mass-weighted average of position vectors of all parts of the object. The object may be viewed as consisting of discrete parts (the first definition), or continuous parts (the 2nd definition), but really these two definitions should be viewed as equivalent, since an integration is essentially a summation.

Note that in the above definitions, the denominator ($\sum_i m_i$ or $\int dm$) is the total mass, which we will denote as M.



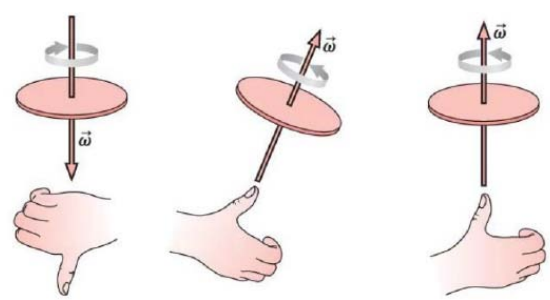
Parallel axis theorem [optional]

The **parallel axis theorem** is a very well-known theorem about the rotational inertia.

$$I = I_{cm} + Md^2 \qquad (14.2)$$

Here, I is the rotational axis around a certain axis (call it axis 1) that does not pass through the center of mass. Then, there must be an axis (call it axis 2) that passes through the center of mass and is parallel to axis 1. I_{cm} is the rotational inertia around axis 2, and d is the distance between axis 1 and axis 2.

Angular velocity, revisited



Note that we defined angular velocity ω as "merely" $d\theta/dt$ thus far. This is OK, but not enough. Why not enough? Consider the case of a uniform circular motion. In this case, the angular velocity ω is constant. What is its direction? It better be constant as well. How do we define its direction? The answer is the right hand rule (see the picture). This is a mere definition, or a convention if you like. Note that the torque vector in terms of position vector and force vector was defined also using the right hand rule. This is no coincidence. With the direction clearly defined like this, we can upgrade our notation of ω to include the arrow symbols as in $\vec{\omega}$.

Recall that we defined a positive angular velocity for a counter clock wise motion and a negative angular velocity for a clock wise motion. What does it really mean? It means the following. As an example, suppose that an object is rotating CCW at an angular speed of 3 Hz in the x-y plane. Then its angular velocity $\vec{\omega} = 3.0\ \hat{z}$ Hz. If it is rotating CW at an angular speed of 3 Hz in the x-y plane, then its angular velocity $\vec{\omega} = -3.0\ \hat{z}$ Hz. Please convince yourself of this by drawing x-y-z axes, and considering the right hand rule.

Angular acceleration, revisited

How about the angular acceleration $\alpha = d\omega/dt$? It is simple. Now that we completely defined the vector $\vec{\omega}$, we can simply upgrade this definition of the angular acceleration to

$$\vec{\alpha} = d\vec{\omega}/dt \qquad (14.3)$$

So, if the angular *speed* (that is, $|\vec{\omega}|$) is increasing, then $\vec{\alpha}$ points parallel to $\vec{\omega}$, i.e. points to the same direction as $\vec{\omega}$. If the angular speed is decreasing, then $\vec{\alpha}$ points anti-parallel to $\vec{\omega}$, i.e. points opposite to the direction of $\vec{\omega}$.

Newton's 2nd law for rotational motion

We already know Newton's 2nd law, and we definitely do not have a new law just because we are considering rotational motions. However, it can be shown (in advanced mechanics) that Newton's 2nd law can be re-written in the following form that is more convenient for a rotational motion.

$$\vec{\tau} = \frac{d\vec{L}}{dt} \qquad (14.4)$$

This equation is valid not only for an inertial reference frame but also for a center of mass reference frame. Here, $\vec{\tau}$ is the *net torque*, i.e. the sum of all torques due to *external* forces. [Actually, $\vec{\tau}$ is the net torque only if all torques due to internal forces cancel each other. This is the case as far as all the materials of this course are concerned, and is a very common case in general anyway. In general, this is the case when internal forces satisfy what is called the "strong form of Newton's 3rd law": not only the two Newton's 3rd law pair forces are equal in magnitude and opposite in direction, but their directions are either parallel or anti-parallel to $\vec{r}_2 - \vec{r}_1$, where $\vec{r}_1, \vec{r}_2$ are position vectors of the two particles exchanging 3rd law pair forces.]

$\vec{L}$ is the **angular momentum**. We will define the angular momentum shortly, but here is one thing to keep in mind.

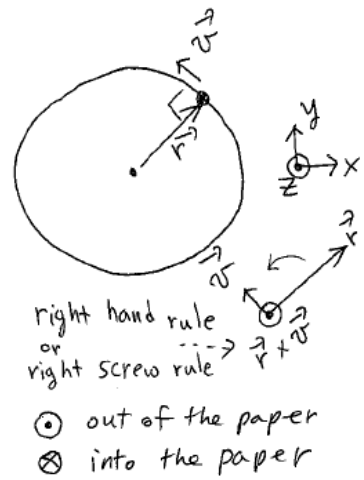
Note that a center of mass reference frame is not an inertial reference frame in general. Think of a figure skater doing a jump or an Olympic diver doing a jump off the board. In both examples, the center of mass is accelerated by the gravitational force and possibly other forces such as air resistance. Thus, the center of mass frame is an accelerated frame. However, by the special nature of the center of mass, the above equation is valid in the center of mass reference frame! [The proof of this fact involves a bit of not-so-complicated vector algebra, and is not usually given at this level. You are welcome to ask me, if you so desire.]

Angular momentum

In general, the angular momentum of a very small particle ("point particle") is defined as

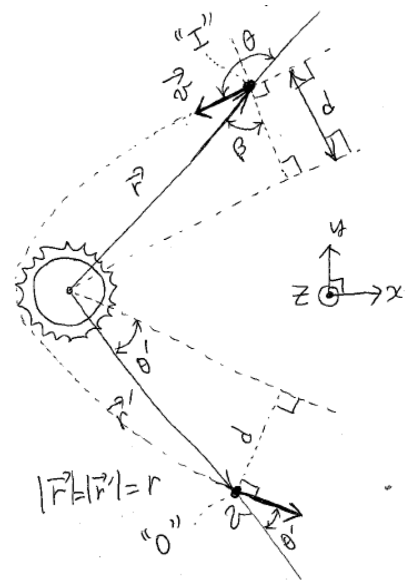
$$\vec{L} = \vec{r} \times \vec{p} = m\vec{r} \times \vec{v} \tag{14.5}$$

Example 1. For a uniform circular motion, show that the angular momentum is constant.



Solution. Consider the CCW motion as depicted in the figure. By the right hand rule (right screw rule may be a better name), $\vec{L}$ points along the z direction. This is true for any position on the circle, not just the one example position shown in the figure. The magnitude of $\vec{L}$ is given by $mrv\sin\frac{\pi}{2} = mrv$, where r and v are the magnitudes of $\vec{r}$ and $\vec{v}$ respectively. r is constant (circular motion), and v is constant (*uniform* circular motion). In conclusion, the magnitude of $\vec{L}$ is constant, as well as its direction, and so $\vec{L}$ is constant. If the motion is CW, then only the direction of $\vec{L}$ is flipped, but the magnitude and the time-independence of $\vec{L}$ remain unchanged.

Example 2. A comet of mass m is observed to approach the Sun and go away from the Sun, as depicted in the figure. The xyz axes are defined as shown. Consider point "I" (incoming) and point "O" (outgoing), for which the distance to the Sun is a common value (r). It is also observed that the two points have a common speed, v, and a common d value (see figure). Show that the angular momentum is identical for points "I" and "O."



Solution. For the "I" point, the angular momentum points to the positive z direction (the right hand rule), and its magnitude is $mrv\sin\theta = mrv\sin(\frac{\pi}{2}+\beta) = mrv\cos\beta = mvd$. For the "O" point, the angular momentum points to the positive direction also, and its magnitude is $mrv\sin\theta' = mvd$. QED. [Note that the angular momentum is constant at any point of motion, actually.]

Angular momentum of a system of particles

$$\vec{L} = \sum_i m_i \vec{r}_i \times \vec{v}_i \ \ \text{(discrete)}, \ \int \vec{r} \times \vec{v}\, dm \ \ \text{(continuous)} \tag{14.6}$$

As always, the continuous case may be viewed also as essentially the discrete case.

Rigid body

A system of many particles whose mutual distances are all constant is called a **rigid body**.

Namely a rigid body is an object whose shape does not change as a function of time, like a sphere, a cylinder, a dumbbell, etc. Note that whether an object is a rigid body or not may also depend on the motion that we consider. Suppose there is a figure skater, who is (a) turning round and round with arms and legs stretched, and then after a while (b) turning round and round with arms and legs pulled in. If we consider (a) and (b), and transitional motions between them, together, the person is definitely not a rigid body. However, if we consider (a) or (b) alone, we may consider the person as a rigid body, since the shape of the body does not change during the motion (a) or (b).

For a rigid body [rotating around a "highly symmetric axis", for the first two], the following equations apply.

$$\vec{L} = I\vec{\omega} \qquad (14.7)$$

$$\vec{\tau} = \frac{d\vec{L}}{dt} = I\,\vec{\alpha} \qquad (14.8)$$

$$K_{rot} = \frac{1}{2} I\omega^2 \ \text{(rotational kinetic energy)} \qquad (14.9)$$

In general, one should be cautious of the fact that $\vec{L} = I\vec{\omega}$ (and thus $\vec{\tau} = I\,\vec{\alpha}$) may not be true. An example would be an apple rotating around an oddly oriented axis, if we take into account the shape of the apple in all details, not approximating it as a simple sphere. In such a case, one can show that $\vec{\omega}$ and $\vec{L}$ are not parallel to each other! But, don't worry. In this course, we won't consider such difficult cases, when we do *quantitative* problems. So, you can assume that the above equations to be valid for any rigid bodies for the purpose of this course. Nevertheless, you should keep in mind that Eqs. (14.4), (14.5), and (14.6) are the more fundamental ones.

Note that the expression for K_{rot} (Eq. (14.9)) is a general one, and is valid even if the rotation axis is not a "highly symmetric axis."

Total kinetic energy for a rigid body

Assume that a rigid body is moving as a whole, while it is also rotating around its center of mass. Let the total mass of the rigid body be M and the velocity of the center of mass be $\vec{V}_{cm}$. Assume that the moment of inertia (just another name for the rotational inertia) around the center of mass is I_{cm} and the angular velocity is $\vec{\omega}$. Then the total kinetic energy is given by

$$K_{total} = \tfrac{1}{2} M V_{cm}^2 + \tfrac{1}{2} I_{cm}\,\omega^2 \qquad (14.10)$$

Keep in mind the importance of the center of mass point in this formula.

Example 3. Suppose an object with the total mass M and the radius R has the rotational inertia $I_{cm} = \gamma M R^2$. For example, $\gamma = 1$ (thin ring or a hollow cylinder), 2/5 (solid sphere), 1/2 (disk or solid cylinder)

2/3 (hollow sphere) are possible values (see Table 10.2 of text, copied in LN 13). Suppose the object is rolling on a surface without slipping. What is the total kinetic energy in terms of M, V_{cm}, and γ?

Solution: For a rolling motion without slipping, we learned that $V_{cm} = R\omega$ (LN 09, page 9; your note should show more details, which I gave in lecture). Thus, the rotational kinetic energy is given by $\frac{1}{2} I_{cm}\omega^2 = \frac{1}{2}(\gamma M R^2)\left(\frac{V_{cm}}{R}\right)^2 = \frac{\gamma}{2} M V_{cm}^2$. Thus, the total kinetic energy = $\frac{1+\gamma}{2} M V_{cm}^2$.

Example 4. A solid sphere and a hollow sphere are released from rest on an incline from the same height. They roll down. Which one arrives at the bottom of the incline first?

Solution: Solid sphere. Why? Because γ is smaller for it than the hollow sphere. Here are the details. Using energy conservation and the result of the previous example, we get $Mgh = \frac{1+\gamma}{2} M V_{cm}^2$, and so $V_{cm} = \sqrt{\frac{2gh}{1+\gamma}}$, where h is the initial height. This means a larger value of V_{cm} for a smaller value of γ. Also, note that V_{cm} does not depend on the mass at all. So, the average speed of the solid sphere must be greater, which means less time for the solid sphere.

"Translation table" (for quick reference only; know physics)

General or linear motions (inertial frame)	**Rotational motion (inertial frame or center of mass frame)**
$\vec{p} = m\vec{v}$ (linear momentum)	$\vec{L} = m\vec{r} \times \vec{v}$ (angular momentum)
$\vec{F}$ (force)	$\vec{\tau} = \vec{r} \times \vec{F}$ (torque)
$\vec{F} = d\vec{p}/dt$ (net force here)	$\vec{\tau} = d\vec{L}/dt$ (net torque here, usually; cf. discussions after Eq. (14.4))
Linear motion in one dimension	**Rotational motion of a rigid body (for L and τ around a "high symmetry axis"; mostly the case in 6A)**
x	θ (angular displacement)
v	ω or $\vec{\omega}$ (see "Angular velocity, revisited")
a	α or $\vec{\alpha}$ (see "Angular acceleration, revisited")
m (inertial mass)	I (rotational inertia or moment of inertia)
$p = mv$	$L = I\omega$ (know to which direction L and ω point!)
$K = \frac{1}{2}mv^2$	$K_{rot} = \frac{1}{2}I\omega^2$ (rotational kinetic energy only)
$F = ma$	$\tau = I\alpha$ (know to which direction τ and α point!)
$dW = Fdx$	$dW = \tau d\theta$ ($W_{net} = \Delta K_{rot}$ for pure rotation, ΔK_{total} in general)

Angular momentum conservation

When the net torque is zero, then the angular momentum is conserved. This follows from $\vec{\tau} = \frac{d\vec{L}}{dt} = 0$.

Linear momentum conservation (obviously not for rotational motion!; included for comparison with the angular momentum conservation)

When the net force is zero, then the linear momentum is conserved. This follows from $\vec{F} = \frac{d\vec{p}}{dt} = 0$.

Although these two extremely important conservation principles are easy to state, one should keep in mind that these conservation principles apply even when the object under consideration consists of many parts and is *not* a rigid body. The above Examples 1 and 2 are simple cases of the $\vec{L}$ conservation at work. Examples 11.2 and 11.3 of textbook are also very important to understand for the angular momentum conservation principle! We will consider the linear momentum conservation in later lectures.