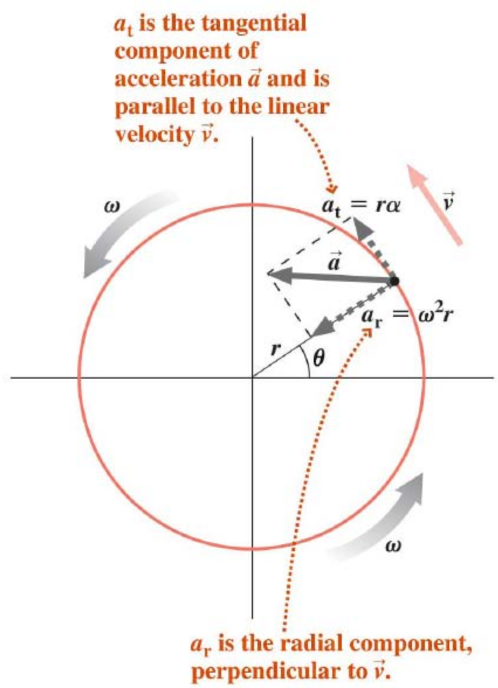
Lecture 13

Rotational Motion

Acceleration in a circular motion



We have discussed many examples of a circular motion so far: the Earth going around the Sun, loop the loop roller coaster, and bug on a head, a pendulum swing, etc. Except for the first example, each of these examples is a *non-uniform circular motion*. What does it mean? It means that the speed (v) is changing while the object is making the circular motion, and so is the angular velocity (ω).

Another name for circular motion is rotational motion.

If one follows the mathematics of LN 5 ("fluxing our calculus muscles"), this time treating ω as a function of time, then one obtains the following. It is not essential to know the math derivation. If you are a math wizard, please try it! Ask me if you need help.

$$v = \omega R \qquad (13.1)$$

$$a_r = R\omega^2 = \frac{v^2}{R} \qquad a_t = R\alpha \qquad (13.2)$$

Note on the notation: I will continue to use R instead of r (as in the book or in the figure on the right borrowed from the book). I find it more appropriate to use R for a constant radius.

Here a_r is the radial acceleration. Another name for it is the centripetal acceleration (a_c), which is exactly the same thing. a_t is the tangential acceleration. What is α? It is the angular acceleration, defined as

$$\alpha = \frac{d\omega}{dt} \quad \text{where} \quad \omega = \frac{d\theta}{dt} \qquad (13.3)$$

Note that **a circular motion, or a rotational motion, is effectively a one dimensional motion**. In what sense? In the above figure, what is changing? Only the angle θ. Only one number. So a one dimensional motion.

In this sense, θ is the *angular displacement*.

Recall that when angle θ is measured counter-clock-wise (CCW), then it is taken as positive. Likewise, when angle θ is measured clock-wise (CW), then it is taken as negative. So, θ can, in principle, take any value from $-\infty$ to ∞, just as the linear displacement x did (LN 03). Lastly, recall that the SI unit of angle is radian (LN 01). [Why do we bother to define θ from $-\infty$ to ∞? Couldn't we just define the range of θ as $-\pi \leq \theta < \pi$ or $0 \leq \theta < 2\pi$ or something like it, stick to it, and be done with it? Yes and no. Strictly speaking, the answer is yes, but practically speaking, it is much more convenient to define angle from $-\infty$ to ∞. If we choose to do the latter, then the motion of the Earth going around the Sun will be starting from a certain value of θ (call that θ_0) and then the θ value increasing steadily and indefinitely. If we had restricted the θ value range to a finite range of 2π, then the same motion will involve a discontinuous jump of the θ value once every year. This is very inconvenient, mathematically speaking.]

Constant angular acceleration motion

Given that θ is the angular displacement, the 1D kinematics machinery of LN 03 and related book parts can be used with just the change of names. In 1D kinematics, we talked about velocity (v), acceleration (a), given the displacement vector x. In constant angular acceleration motion, we talk about angular velocity (ω), angular acceleration (α), given the angular displacement vector θ. Thus for constant angular acceleration α,

$$\omega = \omega_0 + \alpha t, \qquad \theta = \theta_0 + \omega_0 t + \frac{\alpha}{2}t^2 \tag{13.4}$$

$$\frac{\omega_0 + \omega}{2} = \frac{\theta - \theta_0}{t}, \qquad \omega^2 - \omega_0^2 = 2\alpha(\theta - \theta_0) \tag{13.5}$$

Here, the quantity $(\omega_0 + \omega)/2$ corresponds to the average angular velocity ($\bar{\omega} = \frac{\Delta\theta}{\Delta t}$) for time 0 to t.

Examples 10.1 and 10.2 of the textbook should be mastered.

Rotational Inertia

So far, for the most part, we have reduced a physical object with size and shape (such as an apple dropped, a base ball tossed up, a package delivered from an airplane, etc.) to a mere point. How is this possible? The answer is that it is not when rotational motions are involved.

Suppose you consider a rotational motion of an object around a certain axis. The following two forms of the definitions of the **rotational inertia** (I) can be viewed as essentially the same, since, after all, the integration is just a summation over many many little pieces.

$$I = \sum m_i r_i^2 \text{ (discrete) or } \quad I = \int r^2 dm \text{ (continuous)} \tag{13.6}$$

Here, m_i (or dm) is the mass of each little piece ("mass element") that is summed over, and r_i (or r) is the distance from the rotational axis to the mass element.

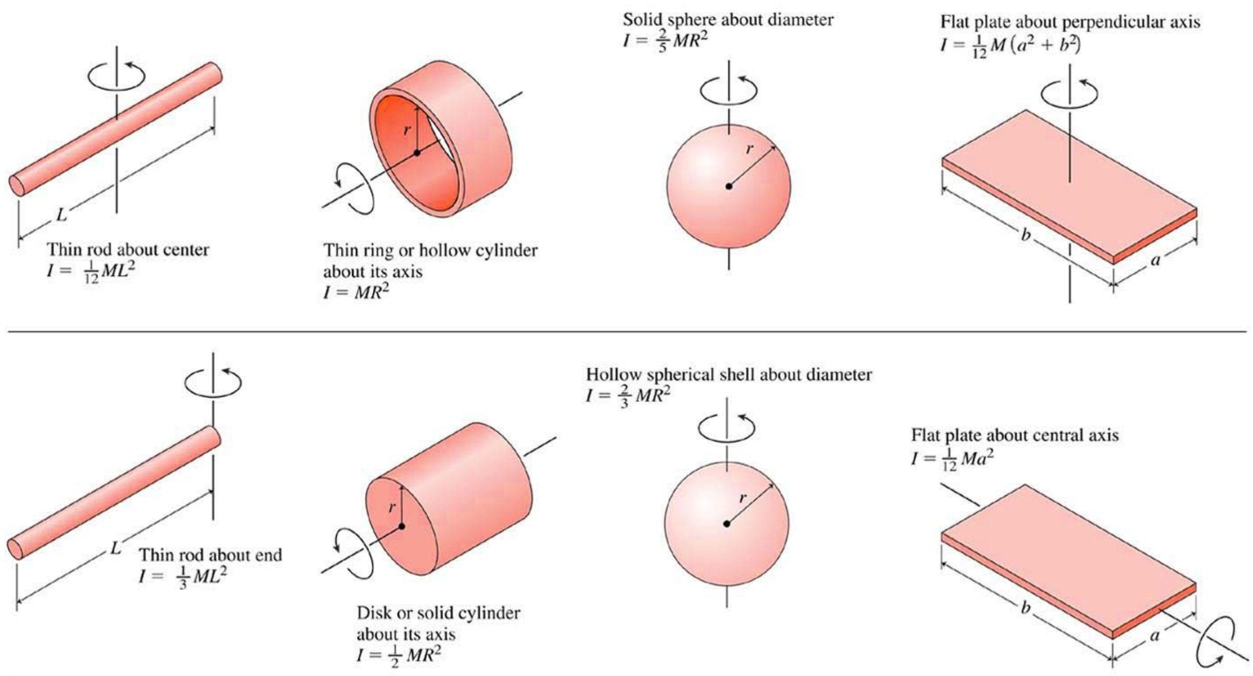
What is the meaning of the rotational inertia?

As a brief digression, let us recall that in $\vec{F} = m\vec{a}$, m is called the *inertial mass* (LN 07). What does it mean? If you hit a baseball with a baseball bat, then the two exchanges forces that are equal in magnitude by Newton's 3rd law. However, since the baseball is much lighter, it is accelerated much more since the acceleration vector is inversely proportional to the mass, $\vec{a} = \vec{F}/m$. Because of this, right after the collision, the baseball can move extremely fast, while the bat does not have any speed nearly that fast. That is, the greater the inertial mass, the harder to accelerate.

The rotational inertia is an equivalent concept in a rotational motion. It is easy to rotate a mass near the axis, while it's harder to do so when the mass is away. Think about a figure skater spinning, while her body is coiled up (fast rotation) or her arms and legs are extended (slow rotation). Or an Olympic diver who coils up (fast rotation) or stretching (slow rotation) before entering water.

Here are some typical geometries and their rotational inertia. You may need to look these up when you do problems. All these formulas can be derived using the multivariable calculus using the above definition, but we won't spend our time on that.

TABLE 10.2 Rotational Inertias



Torque

What is torque? You may have heard about this when you were researching for a car to buy. The reason why we define it is because, for a rotational motion, it is often much more convenient to talk about the torque, $\vec{\tau}$, instead of the force. It is defined as

$$\vec{\tau} = \vec{r} \times \vec{F} \quad \text{(thus, the magniutude is given by } \tau = rF\sin\theta) \qquad (13.7)$$

Here $\vec{r}$ is the position vector at which the force $\vec{F}$ is applied, and θ is the angle between the two vectors. For the definition of the vector product, refer to LN 09.5. Recall that the direction of the vector $\vec{\tau}$ is determined by rotating the first vector ($\vec{r}$) towards the second vector ($\vec{F}$), and applying the right hand rule.

As we will see, the net torque is the cause of an angular acceleration, just as the net force is the cause of an acceleration.