Lecture 12

Conservative force, potential energy, and the mechanical energy conservation

Conservative force and non-conservative force

As the name implies a conservative force is a "good, saving" force, while a non-conservative force is a "bad, wasteful" force. The following mathematical definition applies.

$$\oint \vec{F} \cdot d\vec{r} = 0 \qquad \text{(conservative force)} \tag{12.1}$$

Here, the symbol $\oint$ means an integral over a closed path (any path that begins and ends at the same point; like, but not only, a circle). This definition should read as "$\vec{F}$ **is a conservative force if its integral over <u>any</u> closed path is zero**." It follows that if a force is such that its integral over some closed path is non-zero, then that is not a conservative force, namely, that is a non-conservative force.

Example 1. Show that the gravity (near the surface of the Earth for now; later this will be generalized to anywhere) is a conservative force, as is the spring force.

> *Solution*: From Example 2 of LN 10, the work done by gravity is given by $-mg\Delta y$ for *any* path. It follows that then for any closed path, the work done by gravity is zero, since $\Delta y = 0$ for any closed path. For the spring force, doing like Example 1 of LN 10, the work done by the spring force on an object connected to it is $\int(-kx)dx$. This is equal to $-\frac{1}{2}k\Delta(x^2)$. For any closed path, $\Delta(x^2) = 0$, and so the spring force is a conservative force.

Example 2. Show that the kinetic friction (or the static friction for a wheel) is not a conservative force.

> *Solution*: Consider dragging a big piece of luggage (without wheels – kinetic friction – or with wheels – static friction) around a circle, starting from one point and coming back to it. The friction between the luggage and the floor is always opposite to the velocity of the luggage (and thus opposite to the infinitesimal displacement of the luggage: $d\vec{r} = \vec{v}dt$), and so the work done by the friction is negative for any segment of the circle. It follows then that the work done by the friction is negative for the entire circle. [Note that the static (or the rolling) friction of a wheel is a surprisingly subtle concept. For a full discussion, see http://samphy6a.blogspot.com/2008/11/pardox-of-frictional-force.html.]

Friction is "not a nice force" in the sense that it never does a positive work on an object. It does only negative or zero work, no matter what. It only takes, without ever giving back! [Where does it "waste" the work? Answer: to the environment as heat.] However, the gravity or the spring force "gives back." In a portion of a closed path, it may do a negative work on an object, but, if it did, then in the rest portion of the path, it does a positive work that exactly compensates that negative work, so the total work over a closed path is always *precisely* zero.

Potential energy

For a conservative force, $\vec{F}$, its potential energy function $U(\vec{r})$ is defined as:

$$\Delta U = U(\vec{r}_2) - U(\vec{r}_1) = -\int_{\vec{r}_1}^{\vec{r}_2} \vec{F} \cdot d\vec{r} \tag{12.2}$$

As in Example 2 of LN 10, imagine that you are moving the object very very slowly ("adiabatically"; thus $\vec{a} = 0$) exactly cancelling the conservative force $\vec{F}$ (e.g. gravity or the spring force) with your force = $-\vec{F}$ (for simplicity, assume there is no other force than these two forces). It is in this sense that **the potential energy change is the work done against the conservative force**. Note also that the integral in (12.2) is for *any* path connecting two the end points $\vec{r}_1$ and $\vec{r}_2$. Due to the property of the conservative force (12.1), $\int_{\vec{r}_1}^{\vec{r}_2} \vec{F} \cdot d\vec{r}$ depends *only* on end points $\vec{r}_1$ and $\vec{r}_2$. Lastly, note that (12.2) only defines the difference of $U(\vec{r})$. Thus, this definition leaves the function $U(\vec{r})$ ambiguous up to an arbitrary offset: if $U(\vec{r})$ is a function that satisfies (12.2), then $U(\vec{r}) + C$ also satisfies (12.2) for any constant energy value C. Physically, this means the following. It does not matter where you take the zero of the potential energy $U(\vec{r})$. What matters it the change of $U(\vec{r})$. *Just like the absolute value of the velocity, the absolute value of the potential energy function is absolutely meaningless*. For a given problem, take the zero of the potential energy as conveniently as possible. Of course, once you choose the zero, you have to stick with it within that problem! Mathematically, you can say that $U(\vec{r}_1)$ is an indefinite integral, not a definite integral, so it is arbitrary up to a constant.

Example 3. **Gravitational potential energy near the surface of the earth**. $U(\vec{r}) = mgy$, if the y axis is taken to be pointing up. [cf. Example 7.1 of textbook]

> *Solution*. The gravitational force is $\vec{F} = -mg\hat{\jmath}$. Using (12.2), we have $\Delta U = mg\Delta y$. So
>
> $$U(\vec{r}) = mgy \tag{12.3}$$
>
> up to a constant offset. It is customary, but not required in any way, to take that constant offset to be zero.

Example 4. **Elastic potential energy of a spring**. Consider a spring fixed at one end, like shown in Example 1 of LN 10. Show that $U(x) = \frac{1}{2}kx^2$. [cf. Example 7.2 of textbook]

> *Solution*. $F = -kx$ (Hooke's law). Using (12.2), we have $\Delta U = \frac{1}{2}k\Delta(x^2)$. Thus,
>
> $$U(x) = \frac{1}{2}kx^2 \tag{12.4}$$
>
> up to a constant offset. It is customary, but not required in any way, to take that constant offset to be zero. *This equation is also valid when both ends of the spring are free to move*. In that case x should be interpreted as the change of the distance by which the spring is compressed or elongated compared to its natural length (cf. LN 08).

Example 7.3 of textbook is optional.

Conservation of mechanical energy

Suppose that all forces acting on an object are conservative forces. Then, the mechanical energy of the object, defined as $K + \sum_i U_i(\vec{r})$ where $K = mv^2/2$ is the kinetic energy and U_i is the potential energy for the i-th force, is conserved. Namely, for any point 1 $(t_1, \vec{r}_1, \vec{v}_1)$ and 2 $(t_2, \vec{r}_2, \vec{v}_2)$ of the motion:

$$\frac{1}{2}mv_1^2 + \sum_i U_i(\vec{r}_1) = \frac{1}{2}mv_2^2 + \sum_i U_i(\vec{r}_2) \tag{12.5}$$

The total mechanical energy

$$E = K + \sum_i U_i(\vec{r}) \tag{12.6}$$

is an example of a "**constant of motion**," something that never changes during the course of the motion, however complicated the motion may be.

The proof of this conservation principle is very simple. From the work-energy theorem, $K_2 - K_1 = \int_{\vec{r}_1}^{\vec{r}_2} \vec{F} \cdot d\vec{r}$, where $\vec{F}$ is the net force = $\sum_i \vec{F}_i$. Thus, $K_2 - K_1 = \sum_i \int_{\vec{r}_1}^{\vec{r}_2} \vec{F}_i \cdot d\vec{r}$. By assumption, each $\vec{F}_i$ is a conservative force, and so $\int_{\vec{r}_1}^{\vec{r}_2} \vec{F}_i \cdot d\vec{r} = U_i(\vec{r}_1) - U_i(\vec{r}_2)$ from (12.2). Thus, we have $K_2 - K_1 = \sum_i [U_i(\vec{r}_1) - U_i(\vec{r}_2)]$, which means $K_1 + \sum_i U_i(\vec{r}_1) = K_2 + \sum_i U_i(\vec{r}_2)$, completing the proof.

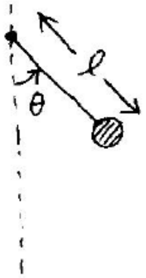
Examples 7.4, 7.5, and 7.6 of the textbook should be mastered thoroughly.

Force from the potential energy

$$F = -dU/dx \quad \text{(in 1D)} \tag{12.7}$$

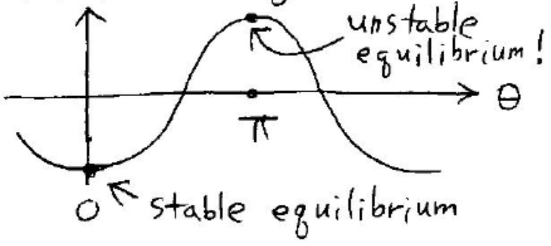
$$\vec{F} = -\nabla U \quad \text{(in general)} \tag{12.8}$$

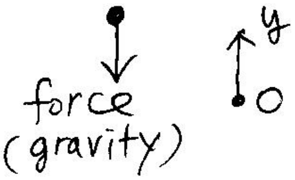
[If you are not familiar with the gradient operator (∇ in the 2nd form), don't worry. In the following, just replace "gradient" with "derivative," and everything will remain valid. We will be dealing with 1D cases only in 6A, anyway.] So, in general, a conservative force is given as the negative gradient of the potential function. Note that the gradient of the potential function is the direction along which the potential function increases *optimally*. [What does this mean? It means the following. At any given point $\vec{r}$, consider a very small circle around it. Consider all possible displacements from $\vec{r}$ to a point on that circle. Which displacement makes the function U increase by the greatest amount? Ans: the displacement vector that has the same direction as the gradient vector of U at $\vec{r}$. Which displacement makes the function U decrease by the greatest amount? Ans: the displacement vector that is opposite to the direction of the gradient vector of U at $\vec{r}$.] Since the force points opposite to the gradient of the potential function, what it means is that **a conservative force is always in the direction to minimize the potential energy function**. This is a very (times 1000) important fact. Note that if the derivative of U is zero, then the force is zero. Such a point is called an equilibrium point. It is a **stable equilibrium** point if the derivative of U is zero at a minimum point of U. It is an **unstable equilibrium** point if the derivative of U is zero at a maximum point of U.



Simple pendulum

$$U(\theta) = -mg\ell\cos\theta$$

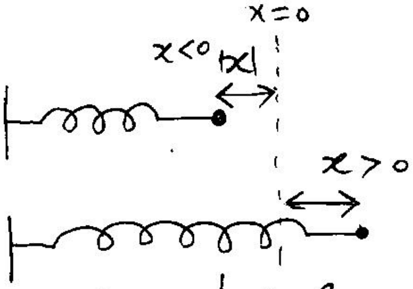




$$U(y) = mgy$$

force decreases U

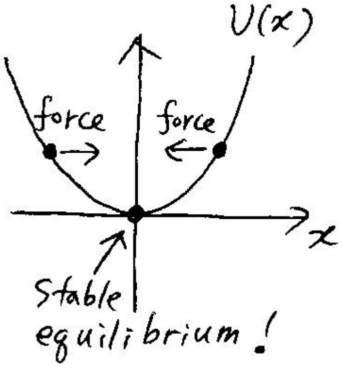
$$F = -\frac{dU}{dy} = -mg$$



$$U(x) = \frac{1}{2}kx^2$$

force decreases U

$$F = -kx$$



Examples 7.7 of the textbook should be mastered.