Lecture 11

Power, Kinetic Energy, and the Work-Energy theorem

Power

An adult is generally a better worker than a child. For example, for a given time an adult usually can do more work than a child. Thus how much work can be done per unit time is an important concept, and this is the concept of power. In previous lectures, we introduced the notations $\vec{F}_{AB}$ and W_{AB}, to mean force by A on B or work by A on B. **The (instantaneous) power supplied/given by A to B** is defined as

$$P_{AB} = \frac{dW_{AB}}{dt} = \vec{F}_{AB} \cdot \vec{v}_B \tag{11.1}$$

Note that $dW_{AB} = \vec{F}_{AB} \cdot d\vec{r}_B$ (this is from (10.1), with explicit A,B subscripts) and thus we get $P_{AB} = \vec{F}_{AB} \cdot d\vec{r}_B/dt = \vec{F}_{AB} \cdot \vec{v}_B$, which is the derivation of the 2nd part of this equation. The **average power** is given by

$$\overline{P}_{AB} = \frac{\Delta W_{AB}}{\Delta t} \tag{11.2}$$

The **SI unit of power** is W(att) = J(oule)/s(econd).

Since the power is the derivative of work, the inverse relation is

$$W_{AB} = \int_{t_1}^{t_2} P_{AB}\, dt \tag{11.3}$$

Kinetic energy

Here we introduce, for the first time, the very important concept: "**energy**." One form of energy is the kinetic energy. Consider the **net force** $\vec{F}$ acting on object B, and consider the power put into the object B by the net force $\vec{F}$ (i.e. the power put into object B by all objects exerting force on B): $P_{net,B} = \vec{F} \cdot \vec{v}_B$. By Newton's 2nd law (we consider constant mass), $\vec{F} = m_B\, \vec{a}_B$, and thus, $P_{net,B} = m_B\, \vec{a}_B \cdot \vec{v}_B$. At this point, note that $\frac{d(\vec{v}_B \cdot \vec{v}_B)}{dt} = \frac{d\vec{v}_B}{dt} \cdot \vec{v}_B + \vec{v}_B \cdot \frac{d\vec{v}_B}{dt} = 2\frac{d\vec{v}_B}{dt} \cdot \vec{v}_B = 2\vec{a}_B \cdot \vec{v}_B$, where the chain rule of the differential calculus and $\frac{d\vec{v}_B}{dt} = \vec{a}_B$ is used. Thus, we get $\vec{a}_B \cdot \vec{v}_B = \frac{d}{dt}\left(\frac{\vec{v}_B \cdot \vec{v}_B}{2}\right) = \frac{d}{dt}\left(\frac{v_B^2}{2}\right)$. Plugging this result into the equation for $P_{net,B}$, we get $P_{net,B} = m_B \frac{d}{dt}\left(\frac{v_B^2}{2}\right) = \frac{d}{dt}\left(\frac{1}{2} m v_B^2\right)$. The quantity $\frac{1}{2} m v_B^2$ is called the **kinetic energy** of B. In general,

$$K = \frac{1}{2} m v^2 \qquad \text{(kinetic energy)} \tag{11.4}$$

Note that v is the speed, i.e. the magnitude of the velocity, in this expression. In terms of this kinetic energy function, what we just proved is extraordinary: $P_{net,B} = \frac{dK_B}{dt}$. Using this on

(11.3), we can see that the work done by the net force on B, $W_{net,B}$, is given as $W_{net,B} = \int_{t_1}^{t_2} P_{AB}\,dt = \int_{t_1}^{t_2} \frac{dK_B}{dt}\,dt = K_{B,2} - K_{B,1} = \Delta K_B$. This is the celebrated work-energy theorem. What it says is that the total work received by an object shows up as the change in its kinetic energy.

Work-Energy theorem

$$W_{net}\,(t_1 \rightarrow t_2) = K_2 - K_1 = \Delta K \qquad \text{(kinetic energy)} \qquad (11.5)$$

The net work received by an object, i.e. the work done by the net force on the object, is equal to the change of its kinetic energy.

The importance and the generality of this theorem cannot be over-emphasized. It is valid for *any* type of motion and *any* type of forces. In a way, this theorem is more general than the more famous "conservation of the mechanical energy," which we will learn shortly. Why? As we will see later, the energy conservation theorem holds only for "nice" forces, i.e. when all forces are "conservative" forces (to be defined later). On the other hand, **the work-energy theorem is always applicable** [in 6A], having no such restriction! You should always remember this important point!

By the work-energy theorem, it follows that the dimension of energy is the dimension of work. Thus, **the unit of energy** is also J(oule). In everyday situations, kW h (kilo-watt-hour) is also an often used unit of energy [1 kW h = 3.6 million J]. Next time you look at your PG&E bill, look for that unit – I guarantee you can find it.

Example 1. Consider our familiar example of tossing a ball, with the initial velocity v_0 (taking "up" to be "positive") and the height h. Ignore air resistance. We consider the symmetric motion of the ball going up and then coming down to the initial position. What is the net work done on the ball (i) on the way up, (ii) on the way down, and (iii) for the total motion?

> *Solution*: (i) On the way up, the final kinetic energy is 0, and so the net work is $\Delta K = 0 - \frac{1}{2}mv_0^2 = -\frac{1}{2}mv_0^2$. (ii) On the way down, $\Delta K = \frac{1}{2}mv_0^2$, and this is the net work done on the ball. (iii) For the total motion, the kinetic energy did not change, and thus the net work done is 0. Note that in this example the only force acting on the ball is the gravity, and the work done by it is negative for (i) and positive for (ii), and so our answers make sense in terms of the sign. In fact, there is more. We know that the work done by gravity is given by $-mg\Delta y$ (Example 2 of LN 10). So, for part (i) we have $-mgh = -\frac{1}{2}mv_0^2$ and for (ii) we have $-mg(-h) = \frac{1}{2}mv_0^2$. This means $h = \frac{v_0^2}{2g}$, which is what we already know. The "magic" here is that we could find this out without using any equations for $x(t)$ and $v(t)$. *This is often the case – when you are not asked about time or velocity components, the work-energy theorem (or the mechanical energy conservation) is generally a more powerful and quicker way to get the answer.*

Example 2. For a one dimensional constant acceleration problem, show that the equation $v^2 - v_0^2 = 2a(x - x_0)$ follows directly from the work-energy theorem.

Solution: The constant acceleration means that the net force is constant also, due to $F = ma$ (assuming constant m of course). In this case the net work is simply $F\Delta x$, from (10.4). By the work energy theorem, $F\Delta x = \Delta K = \frac{1}{2}m(v^2 - v_0^2)$. On the other hand, $F\Delta x = ma(x - x_0)$. From these two equations, we get $2a(x - x_0) = v^2 - v_0^2$.

Examples 6.6, 6.7 (basic), 6.8 (basic), and 6.9 of the textbook must be mastered.