Lecture 10

Work

Work, infinitesimal

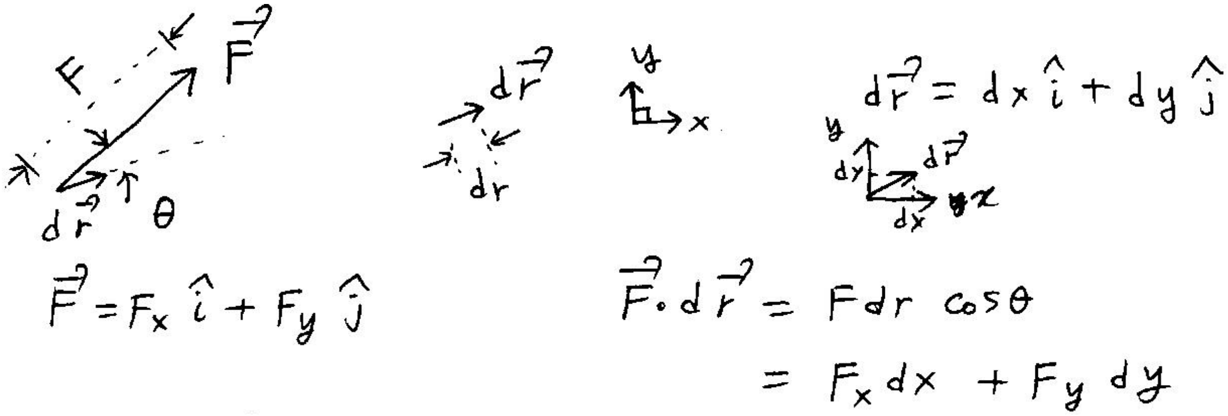
Suppose that a force $\vec{F}$ (which may or may not be a net force) is acting on an object. We consider an infinitesimal time interval, dt, during which the position changes by the displacement, $d\vec{r}$. The infinitesimal work, dW, done by the force $\vec{F}$ on the object is defined as

$$dW = \vec{F} \cdot d\vec{r} \tag{10.1}$$

In mathematics, “infinitesimal” means something like “smaller than any number that you can think of.” In physics, it simply means “very small compared to the scale of the problem or the measurement error.” One thing to remember is that as far as these infinitesimal quantities are concerned, *any* motion is a motion on an approximately straight line (“tangent”). This is because, as I discussed in the mini-review of calculus a while ago, any curve is a line when looked at in very small length scales.

Note that we are using the scalar product here. If not familiar with it, you need to go back to LN 04. If the angle between $\vec{F}$ and $d\vec{r}$ is θ, $\vec{F} = F_x\hat{\imath} + F_y\hat{\jmath}$ and $d\vec{r} = dx\hat{\imath} + dy\hat{\jmath}$,

$$dW = \vec{F} \cdot d\vec{r} = F dr \cos\theta = F_x dx + F_y dy \tag{10.2}$$

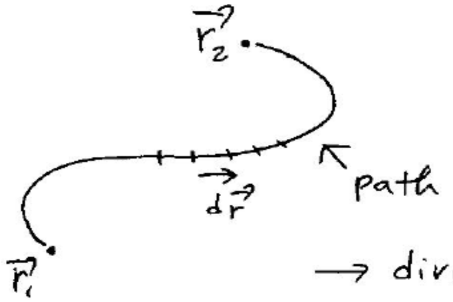


Work, finite

Suppose a force $\vec{F}$ (which may or may not be a net force) is acting on an object, which moves from position $\vec{r}_1$ to $\vec{r}_2$. The work done by the force $\vec{F}$ on the object is then given by

$$W = \int_{\vec{r}_1}^{\vec{r}_2} \vec{F} \cdot d\vec{r} \tag{10.3}$$

The integral expression is called a "line integral". You need not be scared by it, even if you did not learn about it. It is the sum over all infinitesimal dW's corresponding to a "gazillion" little paths that make up the entire path when summed up. [Note: we use subscript1 (as in $t_1, \vec{r}_1, \vec{v}_1$) and subscript 2 (as in $t_2, \vec{r}_2, \vec{v}_2$) to mean two points of interest in a motion.]



$$W = \int_{\vec r_1}^{\vec r_2} \vec F \cdot d\vec r$$

What does it mean?

→ divide the path into a gazillion little paths

$dW = \vec F \cdot d\vec r$ for each path

→ sum up all dW's

→ $W = \int_{\vec r_1}^{\vec r_2} \vec F \cdot d\vec r$

Note that in general $\vec{F}$ changes as a function of position so it cannot be taken out of the integral symbol. If it is known to be a constant value, then it can. In that case,

$$W = \vec{F} \cdot \int_{\vec{r}_1}^{\vec{r}_2} d\vec{r} = \vec{F} \cdot (\vec{r}_2 - \vec{r}_1) = \vec{F} \cdot \Delta\vec{r} \quad (constant \text{ force}) \qquad (10.4)$$

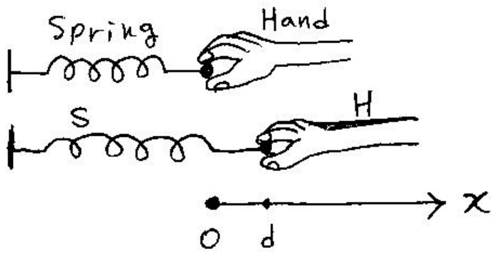
Note in all of this that we talk about the work done by a force, not necessarily the net force, when an object is moving from one place to the next. The motion of an object is always the result of the net force by Newton's 2nd law. For that motion, we can talk about the work done by various forces. Some may have done positive work, and some may have done negative work, and some may have done no work! This all depends on the angle between the force and the displacement! **Just as $\vec{v}$ and $\vec{a}$ are "apple and orange," so are $\vec{F}$ and $d\vec{r}$ (or $\Delta\vec{r}$)!**

The SI unit of work = Joule

J(oule) is the SI unit of work, in other words, J = N m (i.e. newton-meter) = kg m^2/s^2. As we will see later, the unit of work is the same as the unit of energy (as is the unit of heat). 4.2 Joules is 1 calorie.

Example 1: A spring is attached to a wall. The spring constant k = 100 N /m. You grab the other end, and you stretch the spring very very slowly ("adiabatically"). You stretch it by 1 cm = 0.01 m. How much work did you do to the spring? How much work did the spring do to you?

> *Solution*: W_{HS} (work done by hand on spring) = 0.005 J. W_{SH} = –0.005 J. See the figure for explanation. Why is the work done by spring on hand negative? Because the spring was exerting the force in the opposite direction to the movement. In contrast, F_{HS} did a positive work, since its direction was parallel to the direction of the motion. [Note that in this example x *is* the position coordinate of an object (end of spring), since the other end of the spring is fixed (cf. LN 08).]



Position of spring end
= position of hand
("finger tips")
= x $(0 \longrightarrow d)$

Force by spring on hand $F_{SH} = -kx$
(Hooke's law)

Force by hand on spring $F_{HS} = kx$ (Newton 3rd)

Work by hand on spring

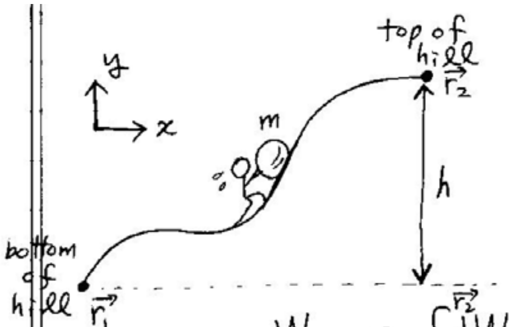
$$: W_{HS} = \int_0^d F_{HS}\,dx = \int_0^d kx\,dx = \frac{1}{2}kx^2\Big|_0^d = \frac{1}{2}kd^2$$

Work by spring on hand

$$: W_{SH} = \int_0^d F_{SH}\,dx = \dots \quad \dots = -\frac{1}{2}kd^2$$

Example 2: Sisyphus is rolling a boulder from the bottom of the hill to the top of the hill, as shown. What is the work that the Earth('s gravity) did on the boulder?

Solution: As shown in the figure, the work by the Earth on the boulder is $-mgh$ where m is the mass of the boulder, and h is the height of the hill. ***Note that this solution does not depend on the shape of the hill at all!*** Also, note that the Earth did a negative work on the boulder. [Poor Sisyphus of course must have done a positive work on the boulder, because he pushed the ball up, i.e. he pushed the ball in the same direction as the direction of the movement. If the ball moved very very slowly (in other words "adiabatically"), then at every moment the acceleration of the ball can be ignored ($\vec{a} = 0$), and so one can show that the work done by Sisyphus is exactly mgh. This is called the *work done against the gravity*. The easiest way to derive this is by using the "work-energy theorem" (next lecture), but you can also show this using the free body diagram of the boulder. I challenge you to do the latter!]



$$\vec{F}_{EB} = -mg\hat{j}$$

earth, boulder

m : mass of boulder

$$d\vec{r} = dx\,\hat{i} + dy\,\hat{j}$$

$$dW_{EB} = -mg\,dy$$

$$W_{EB} = \int_{\vec{r}_1}^{\vec{r}_2} dW_{EB} = -mg\int_{\vec{r}_1}^{\vec{r}_2} dy = \boxed{-mg\,\Delta y}$$

$$= -mgh$$

$$\boxed{W_{EB} = -mgh}$$

Examples 6.1 (basic), 6.2, 6.3, 6.4 of the textbook must be mastered.