

Lecture 6

Reference frame and projectile motion

Reference frame

We have already used reference frames, and the importance of them cannot be over-emphasized. To describe motions, we use a spatial coordinate system to quantify positions in space. This coordinate system is called a **reference frame**. Since the position vector (\vec{r}), the starting point of kinematics, *cannot even be defined* without a reference frame, one can immediately recognize the importance of the reference frame. In general, the best reference frame is the one that makes the description of the motion of interest as concise as possible. Often this requires the choice of a **moving reference frame**.

The reference frame of someone/something means a reference frame in which that someone/something is at rest. For instance, the reference frame of a runner means a coordinate system whose origin is moving exactly like the runner, and the reference frame of an air current means a coordinate system whose origin is moving exactly like the air current. In these usages, the expression “reference frame” can be replaced by the expression “rest frame.”

Example 1. On a straight road, you are driving in a car A at velocity 40 mph. There is a car B in front of you, and the distance between that car and your car is increasing at 20 mph. In your reference frame, what is the velocity of car B (call this v_{BA})? What is the velocity of your car A in the reference frame of car B (call this v_{AB})? In the reference frame of the road, what is the velocity of car B (call this v_{BR}) and the velocity of car A (call this v_{AR})?

Solutions: From your point of view, car B moving at 20 mph, so $v_{BA} = 20$ mph. On the other hand, the person in car B will see your car as moving backwards at the same speed, and $v_{AB} = -20$ mph. v_{AR} was given as 40 mph. Since car B is going faster by 20 mph, $v_{BR} = 60$ mph.

Simple as it is, this example points to a very valuable vector relation: $v_{BA} = v_{BR} - v_{AR} = -v_{AB}$. This relation is immediately generalized to any dimensions:

$$\vec{v}_{BA} = \vec{v}_{BR} - \vec{v}_{AR} \quad (6.1)$$

This relation, valid for *any* three objects A,B,R in *any* type of motion, can be read as: the velocity of the object B in the reference frame of A is the velocity of the object B in the reference frame of R minus the velocity of the object A in the reference frame of R. Or, since $\vec{v}_{AR} = -\vec{v}_{RA}$, this equation can be rewritten as a sort of “chain-rule” for velocity:

$$\vec{v}_{BA} = \vec{v}_{BR} + \vec{v}_{RA} \quad (6.2)$$

Relative velocity

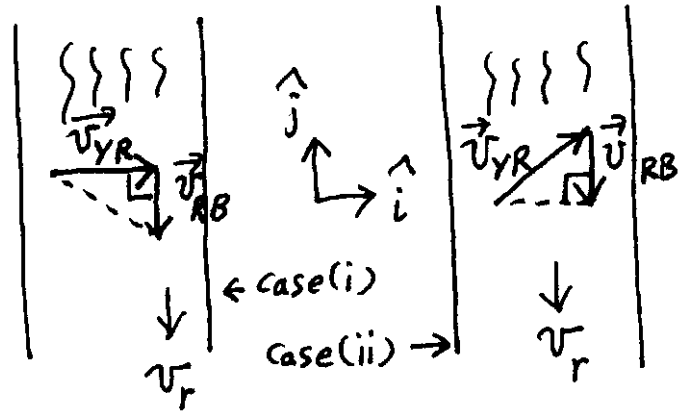
\vec{v}_{BA} is also defined as a relative velocity: it is the velocity of B relative to A.

Examples 3.2 of the text should be mastered.

Example 2. A straight river (R) is flowing at 0.9 m/s towards south. You (Y) can swim as fast as 1.2 m/s. You need to cross the river to get to the other bank (B). In one scenario (i) you decide to swim east within the river's reference frame, and in the other scenario (ii) you decide to swim east within the bank's reference frame. Calculate the relative velocity vector \vec{v}_{YB} and the speed $|\vec{v}_{YB}|$ in these two cases.

Solutions: The key thing to notice in this problem is that the swimming ability of a person determines the magnitude of the velocity of the person *relative to the river*. This means $|\vec{v}_{YR}| = 1.2$ m/s. [Note that a similar consideration applies to airplanes flying in air current.] Let us take the coordinate system so that the x axis is pointing east and the y axis is pointing north. Then, $\vec{v}_{RB} = -v_r \hat{j}$, where $v_r = 0.9$ m/s. This equation, along with $\vec{v}_{YB} = \vec{v}_{YR} + \vec{v}_{RB}$ and $|\vec{v}_{YR}| = 1.2$ m/s, gives enough information to solve the problem (see figure). Case (i): $\vec{v}_{YR} = 1.2 \hat{i}$ m/s.

Thus, $\vec{v}_{YB} = 1.2 \hat{i} - 0.9 \hat{j}$ m/s. $|\vec{v}_{YB}| = 1.5$ m/s. Case (ii): $\vec{v}_{YR} = \sqrt{1.2^2 - 0.9^2} \hat{i}$ m/s = 0.8 \hat{i} m/s. [So, which is the faster way to cross the river?]



Constant acceleration problem

The following solutions are valid in any dimensions, due to the fact taking the derivative of a vector can be done component-wise as shown in Equation (4.16). When the acceleration \vec{a} is constant,

$$\vec{v} = \vec{v}_0 + \vec{a}t \quad (6.3)$$

$$\vec{r} = \vec{r}_0 + \vec{v}_0 t + \frac{1}{2} \vec{a} t^2 \quad (6.4)$$

$$\frac{\vec{v}_0 + \vec{v}}{2} = \frac{\vec{r} - \vec{r}_0}{t} \quad (6.5)$$

The above equations are direct generalizations of Equations (3.1) through (3.3), and can be readily proven component-wise using them.

Examples 3.3 of the text should be mastered.

Projectile motion

Remember that “free fall” motions are defined as motions caused by the gravitational force alone. Projectile motions are defined as free fall motions with the constant downward surface gravity with magnitude $= g \approx 9.8$ m/s². Then, the projectile motion is a constant acceleration problem with $\vec{a} = -g\hat{j}$, if we take the y axis to point up. Assuming the initial position $\vec{r}_0 = x_0 \hat{i} + y_0 \hat{j}$ and the initial

velocity $\vec{v}_0 = v_{0,x} \hat{i} + v_{0,y} \hat{j}$, the above solutions (6.3) and (6.4) can be expressed in the following component-wise fashion:

$$v_x = v_{x,0} \quad (6.6)$$

$$x = x_0 + v_{x,0}t \quad (6.7)$$

$$v_y = v_{y,0} - gt \quad (6.8)$$

$$y = y_0 + v_{y,0}t - \frac{1}{2}gt^2 \quad (6.9)$$

These equations should be understood as follows. (1) The motion along the x axis and the motion along the y axis are independent. (2) The motion along the x axis is a constant *velocity* motion (due to no acceleration in that direction). (3) The motion along the y axis is a constant acceleration motion.

Note that x is a linear function of t , (i.e. in the form of $At + B$) and thus t is a linear function of x . On the other hand y is a quadratic function of t (i.e. in the form of $At^2 + Bt + C$). It then follows that y is a quadratic function of x , and thus the trajectory of a projectile motion, i.e. y as a function of x , is a parabola. [For advanced students: in fact, we have just proved that the trajectory of *any* non-zero constant acceleration motion in 2D is a parabola.] Recall that we discussed a uniform circular motion in the previous lecture. Obviously a circle is not a parabola, and thus it follows that a circular motion is not a projectile motion! Why not? It is because the acceleration vector is *not* constant for a circular motion, as we learned in (5.11).

This is the end of the theory of projectile motions. Note that the text lists more equations. However, equations (3.14) and (3.15) should NEVER be used ... without showing the derivation. As we shall see below, they really aren't necessary if you know how to apply the above four equations.

One piece of information that is good to remember about a projectile motion is this: **the time to reach the maximum height and the maximum height are determined completely by one parameter $v_{0,y}$, the y component of the initial velocity**. Why? It is for the same reason that the initial speed determined the hang time and the maximum height in the “foul ball” problem (HW). In other words, Equations (6.8) and (6.9) have slightly different symbols than equations that we dealt with in LN 3, but otherwise they are identical! We know how to solve them! Doing just like the “foul ball problem” of homework (or Example 5 of LN 3), we get

$$t_h = v_{0,y}/g$$

and

$$h = v_{0,y}^2/(2g)$$

where t_h is the time it takes to reach the maximum height h .

Example 3. At what angle should you throw a ball to make it go farthest? Assume that you can throw a ball at a certain maximum speed, regardless of direction. Ignore air resistance and the height of the person throwing the ball.

Solutions: Let us call the maximum speed at which the ball is thrown, v_0 . This is also the speed of the ball at $t = 0$. Assume that at a later time t_f , the ball comes back to the ground. Ignoring the height of the person throwing the ball, $t_f = 2t_h = 2v_{0,y}/g$. During this time, the displacement in x is given simply by $d = v_{0,x}t_f = 2v_{0,x}v_{0,y}/g$. How to maximize this range d ? With the usual definition of θ (cf. Figure on page 2 of LN 5), $v_{0,x} = v_0 \cos \theta$ and $v_{0,y} = v_0 \sin \theta$, and thus $d = 2v_0^2 \cos \theta \sin \theta / g$. Using the well-known identity $2 \sin \theta \cos \theta = \sin(2\theta)$, we get $d = v_0^2 \sin(2\theta) / g$. To maximize this as a function of θ , $\theta = \pi/4$ (or 45 degrees), which is the answer.

Example 3.4 of the text should be mastered.

Example 4. (Example 3.5 of text) A construction worker throws a tool like shown in the figure. At what speed should he throw the tool at, in order to clear the edge of the hole?

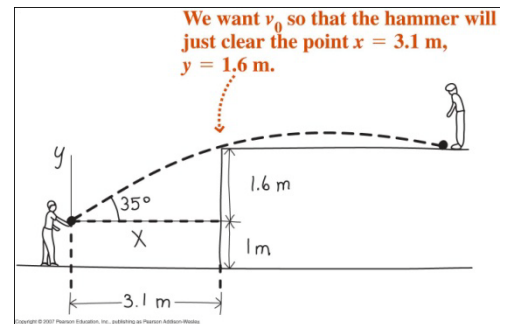
Solutions: Note that the time it takes for the tool to reach the edge of the hole is $t_E = d/v_{0,x}$ where $d = 3.1$ m. During this time, we want to make sure that the change in y is greater than $h = 1.6$ m. The change in y during time t_E is given by $v_{0,y} t_E -$

$\frac{1}{2} g t_E^2 = d \frac{v_{0,y}}{v_{0,x}} - \frac{1}{2} g \frac{d^2}{v_{0,x}^2}$. What we need to do is to set this equal to h , to solve for v_0 , where

$v_{0,x} = v_0 \cos 35^\circ$ and $v_{0,y} = v_0 \sin 35^\circ$. Which means $d \tan 35^\circ - \frac{1}{2} g \frac{d^2}{v_0^2 \cos^2 35^\circ} = h$. Note

that all quantities except v_0 are given in this equation, so we calculate numbers: in the SI unit this

equation reads $2.171 - 70.176 / v_0^2 = 1.6$. So, $v_0 = \sqrt{70.176 / (2.171 - 1.6)} \text{ m/s} = 11 \text{ m/s}$.



Example 3.6 of the text is optional. [Not because it is difficult.]