Lecture 5

2D Kinematics and uniform circular motion

Kinematics in 2D (and any other dimensions)

The equations for 1D kinematics (2.1) through (2.6) are immediately generalized to any spatial dimensions, simply by using explicit vector symbols (and changing, by convention, the position vector symbol from x to r).

Position, Displacement
$$\vec{r},\ \ \Delta\vec{r} \tag{5.1}$$

Velocity
$$\vec{v} = \lim_{\Delta t\to 0}\frac{\Delta\vec{r}}{\Delta t} = \frac{d\vec{r}}{dt} \tag{5.2}$$

Acceleration
$$\vec{a} = \lim_{\Delta t\to 0}\frac{\Delta\vec{v}}{\Delta t} = \frac{d\vec{v}}{dt} \tag{5.3}$$

Average velocity
$$\bar{\vec{v}} = \frac{\Delta\vec{r}}{\Delta t} = \text{time average of } \vec{v} \tag{5.4}$$

Average acceleration
$$\bar{\vec{a}} = \frac{\Delta\vec{v}}{\Delta t} = \text{time average of } \vec{a} \tag{5.5}$$

Speed (scalar)
$$\text{The magnitude of } \ \vec{v} = |\vec{v}| \tag{5.6}$$

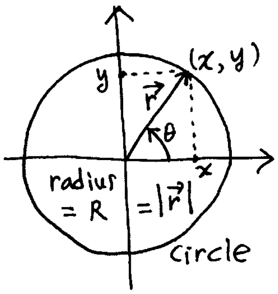
Average speed (scalar)
$$\text{Time average of the instantaneous speed} = \frac{distance}{\Delta t} \tag{5.7}$$

Here, the first three equations are the most fundamental. The first five equations describe vector quantities (position, displacement, velocity, acceleration), and the last two describe scalar quantities (speed, distance).

Equations (2.1) through (2.6) should be considered as special forms of the above equations in the case of 1D. As vector additions, subtractions, and differentiations can be carried out component-wise (Equations (4.13)-(4.16)), the above equations introduce essentially no additional difficulty compared to the 1D form (2.1 through 2.6). One thing to note is the "magnitude." For a vector in any spatial dimensions, the magnitude – the length of the "arrow" – can be shown to be the square root of the sum of squares of all components, by the (repeated) use of the Pythagorean theorem. So, for 2D, speed = $\sqrt{v_x^2 + v_y^2}$ and, for 1D, speed = $\sqrt{v^2} = |v|$.

Example 1. **Uniform circular motion**. The position vector on a circle of radius R is given by (see the figure below):

$$\vec{r} = x\hat{\imath} + y\hat{\jmath} = R(\cos\theta\,\hat{\imath} + \sin\theta\,\hat{\jmath}), \quad \text{i.e., } x = R\cos\theta\,, y = R\sin\theta \tag{5.8}$$



Important reminders (from LN 1 and calculus): (1) We use radians for the unit of the angle, unless specified otherwise. (2) (5.8) is valid for *any* value of θ from $-\infty$ to ∞, not just for, say, the θ values in the first quadrant $0 \le \theta \le \pi/2$. (3) The positive value of θ is measured in the counter-clock-wise fashion, starting from the positive x axis as shown in the figure. The value of θ measured in the clock-wise fashion is defined as negative. (4) Any two values of θ that differ by 2π are equivalent. For example, all the following values of angle are equivalent: $\frac{\pi}{3}, \frac{7\pi}{3}\left(= \frac{\pi}{3} + 2\pi\right), \frac{13\pi}{3}\left(= \frac{\pi}{3} + 4\pi\right), -\frac{5\pi}{3}\left(= \frac{\pi}{3} - 2\pi\right), \ldots$

A uniform circular motion means that the angle is changing at a constant rate (ω):

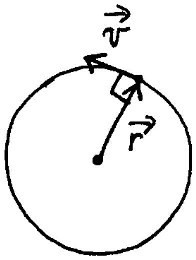
$$\theta = \omega t + \theta_0 \tag{5.9}$$

We just introduced the symbol ω (omega, not double-u) for the **angular velocity**. [While ω is constant for *this* example, its more general definition is $\omega = d\theta/dt$ and ω is then a *function* of time.] Question: what is the acceleration vector $\vec{a}$ for this uniform circular motion?

Solution: First, let us examine what $\vec{v}$ is. From Eq. (5.2), (5.8) and (4.16), we get $\vec{v} = \frac{d\vec{r}}{dt} = \frac{dx}{dt}\hat{\imath} + \frac{dy}{dt}\hat{\jmath}$. From the "chain rule," $\frac{dx}{dt} = \frac{dx}{d\theta}\frac{d\theta}{dt}$. The 2nd term in the product is easy: $\frac{d\theta}{dt} = \omega$ from (5.9). For the 1st term $\frac{dx}{d\theta} = R\frac{d\cos\theta}{d\theta}$. *We measure θ in radians, in which case, and only then,* $\frac{d\cos\theta}{d\theta} = -\sin\theta$. Collecting all these results, we get $\frac{dx}{dt} = -R\sin\theta\,\omega = -\omega y$. Similarly, $\frac{dy}{dt} = \frac{dy}{d\theta}\frac{d\theta}{dt} = (R\cos\theta)\omega = \omega x$. Thus,

$$\vec{v} = R\omega(-\sin\theta\hat{\imath} + \cos\theta\hat{\jmath}) = \omega(-y\hat{\imath} + x\hat{\jmath}) \tag{5.10}$$

At this point, readers should be able to show that $\vec{r}\cdot\vec{v} = 0$, using (4.7), (5.8) [$\vec{r} = x\hat{\imath} + y\hat{\jmath}$] and (5.10) [$\vec{v} = \omega(-y\hat{\imath} + x\hat{\jmath})$]. This means that the two vectors $\vec{r}$ and $\vec{v}$ are orthogonal (or perpendicular) to each other. You should convince yourself that this orthogonality makes a perfect sense for any circular motion (see the right figure). Now that we showed $\frac{dx}{dt} = -\omega y$ and $\frac{dy}{dt} = \omega x$, it is a relatively simple job to take the derivative of (5.10) one more time ($\vec{a} = \frac{d\vec{v}}{dt}$) to get



For *any* motion, the velocity vector is tangential to the trajectory/path. For a circular motion, it is thus orthogonal to the radial direction.

$$\vec{a} = -\omega^2(x\hat{\imath} + y\hat{\jmath}) = -\omega^2\vec{r} \tag{5.11}$$

$$|\vec{a}| = |-\omega^2\vec{r}| = \omega^2 R \tag{5.12}$$

(5.11) is the answer to this question. Notice that the magnitude of the acceleration is constant ($\omega^2 R$ since $|-\vec{r}| = R$), while the direction is *not* constant, being exactly the opposite of the position vector, since $\vec{a}$ is a negative number times $\vec{r}$ (see LN

4, “Multiplying a vector by a scalar”). Thus, the vector $\vec{a}$ is a *function of time*, i.e. in a circular motion *the acceleration vector is* <u>*not*</u> *a constant*. Since by definition $\vec{r}$ points outward from the center of the circle (see the above figure), this means that $\vec{a}$ points *towards* the center. This is why (5.11) is called the **centripetal acceleration** of a circular motion.

Uniform circular motion

The above example gave a nice analysis of the uniform circular motion, directly from the general kinematics equations (5.1) through (5.3). The following equations are also essential to understand for a uniform circular motion.

Angular speed
$$|\omega| = 2\pi/T \tag{5.13}$$

Speed
$$v = |\vec{v}| = R|\omega| \tag{5.14}$$

The magnitude of the centripetal acceleration
$$a = |\vec{a}| = R\omega^2 = v^2/R \tag{5.15}$$

Here, T is the period – the time it takes for one revolution. Since one revolution corresponds to 2π radians, (5.13) is easy to understand: for a uniform circular motion the angular speed is 2π divided by the period. (5.14) can be interpreted as “for a uniform circular motion, the speed is $2\pi R/T$,” or as the consequence of taking the magnitude of (5.10) since $R = \sqrt{x^2 + y^2}$. Finally, (5.15) is directly obtained from (5.12) and (5.14).

Note that Equations (5.10) and (5.11) can be derived using a geometric method as well. Do read the textbook for that derivation, which I will not cover in the lecture in the interest of time.

Examples 3.7 and 3.8 of the text should be mastered.