Lecture 4

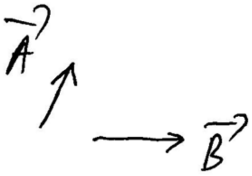
Vectors

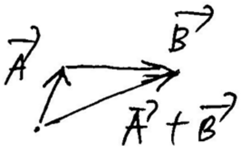
Definition of vector

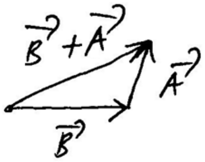
A vector is a mathematical object that is characterized by direction as well as magnitude. This is not a rigorous definition but it is good enough. Visually, it is an arrow, pointing from tail to head. Thus, vector quantities are written with an arrow above them, as in $\vec{F}$ (force) or $\vec{v}$ (velocity).

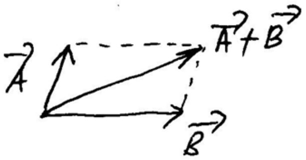
Vector addition

To add two vectors, (1) translate one of the two vectors so that its head is on the tail of the other vector, and (2) draw a new arrow starting from its tail to the head of the other vector. There are two ways to do this, depending on which vector you choose to move, and they give the identical result, i.e., the vector addition is a commutative operation: $\vec{A} + \vec{B} = \vec{B} + \vec{A}$. Another way to add two vectors is (1) bring the tails of the two vectors to a common point, (2) complete a parallelogram starting from the two sides corresponding to the two vectors (see dashed lines above), and (3) draw an arrow from the common tail point of (1) to the new vertex of the parallelogram created in (2).



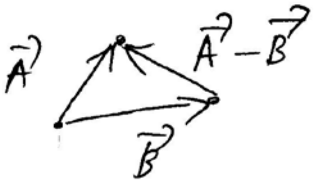






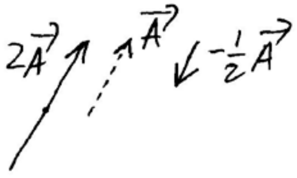
Vector subtraction

The subtraction is the inverse operation of addition. Be sure to understand the example diagram as $\vec{B} + (\vec{A} - \vec{B}) = \vec{A}$.



Multiplying a vector by a scalar

For a given vector $\vec{A}$, $c\vec{A}$, where c is a number (i.e. a scalar), corresponds to scaling that vector. If c is a positive number, then it means stretching or contracting the size of the vector. If c is a negative number, then it means changing the direction of the vector and then stretching or contracting it.

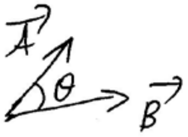


Multiplying two vectors – scalar product

One way to multiply two vectors is to form the so-called the scalar product. The definition is

$$\vec{A}\cdot\vec{B} = AB\cos\theta \tag{4.1}$$

where A (B) means the magnitude of $\vec{A}$ ($\vec{B}$) and θ is the angle between $\vec{A}$ and $\vec{B}$.

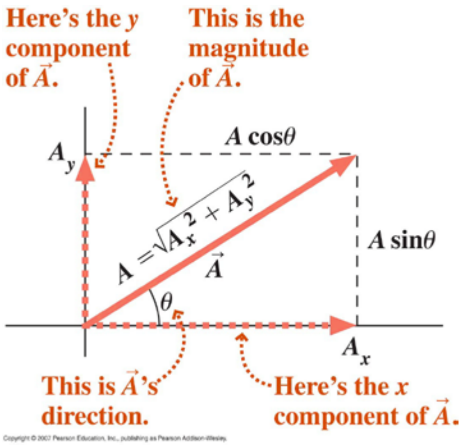


$$\vec{A}\cdot\vec{B} = |\vec{A}||\vec{B}|\cos\theta$$
$$= AB\cos\theta$$

Thus far, we have defined vector addition and two kinds of multiplications. Any combinations of these operations are commutative and associative, just like for ordinary number addition and multiplication.

Components

While the above rules are good for visualizing operations on vectors, they are often cumbersome for *calculations* on vectors. For the latter, we need to *represent* a vector: that is (1) define a coordinate system, (2) project the vector onto the axes of the coordinate system, and (3) read the numbers at the projection points – these numbers are called "components." A collection of components, e.g. (A_x, A_y) for a 2D vector $\vec{A}$, is called a *representation* of $\vec{A}$.



Components as Cartesian coordinates

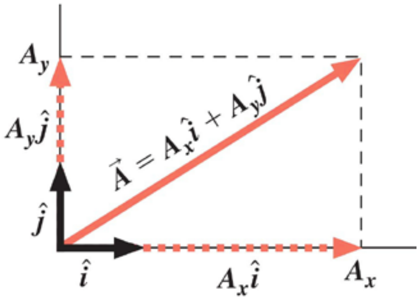
The most basic coordinate system is the Cartesian coordinate system. For this course, it is sufficient to consider a 2D Cartesian coordinate system. This is illustrated in the figure above. The following relations should be *understood thoroughly* (not memorized!) in order to go between "the magnitude, direction description of a vector" and "the component representation of a vector."

$$A_x = A\cos\theta \quad \text{and} \quad A_y = A\sin\theta \tag{4.2}$$

$$A = \sqrt{A_x^2 + A_y^2} \quad \text{and} \quad \tan\theta = A_y/A_x \tag{4.3}$$

Unit vectors

A unit vector for a given axis of the coordinate system is the unit-length vector that points to the positive direction of that axis. Notations $\hat{\imath}, \hat{\jmath}, \hat{k}$ are used to denote unit vectors for x, y, z axes respectively. So are notations $\hat{x}, \hat{y}, \hat{z}$. From the above definition of components, it follows that (for 2D, with a straightforward generalization to any dimensions):



$$A_x = \vec{A}\cdot\hat{\imath},\ A_y = \vec{A}\cdot\hat{\jmath} \tag{4.4}$$

$$\vec{A} = A_x\hat{\imath} + A_y\hat{\jmath} \tag{4.5}$$

Position, displacement, velocity, and acceleration vectors

Any position in space is a vector, in the sense that, once the origin of space is given, one can draw an arrow from the origin to the position. A **position vector** is usually denoted as $\vec{r}$.

Displacement is another **vector** quantity. It is defined as the difference between two position vectors: $\Delta\vec{r} = \vec{r}_2 - \vec{r}_1$.

Here we give the definition of velocity and acceleration vectors in any general dimensions. [That is, the definitions (2.1) through (2.6) are merely special 1D cases of these.]

Average velocity	$\bar{\vec{v}} = \frac{\Delta\vec{r}}{\Delta t}$	(4.6)
Velocity	$\vec{v} = \lim_{\Delta t\to 0}\frac{\Delta\vec{r}}{\Delta t} = \frac{d\vec{r}}{dt}$	(4.7)
Average speed	Time average of the instantaneous speed $= \frac{distance}{\Delta t}$	(4.8)
Speed	$v =$ the magnitude of $\vec{v}$	(4.9)
Average acceleration	$\bar{\vec{a}} = \frac{\Delta\vec{v}}{\Delta t}$	(4.10)
Acceleration	$\vec{a} = \lim_{\Delta t\to 0}\frac{\Delta\vec{v}}{\Delta t} = \frac{d\vec{v}}{dt}$	(4.11)

Note (again) that the three vector quantities $\vec{r}, \vec{v}, \vec{a}$ are the most fundamental quantities here, from which all other quantities can be derived.

Vector calculations in terms of components (or "why components are cool")

For two dimensional vectors $\vec{A} = A_x\hat{\imath} + A_y\hat{\jmath}$ and $\vec{B} = B_x\hat{\imath} + B_y\hat{\jmath}$, the following holds.

$$\vec{A} = \vec{B} \overset{equivalent}{\Longleftrightarrow} A_x = B_x \text{ and } A_y = B_y \tag{4.12}$$

$$\vec{A} + \vec{B} = (A_x + B_x)\hat{\imath} + (A_y + B_y)\hat{\jmath} \tag{4.13}$$

$$\vec{A} - \vec{B} = (A_x - B_x)\hat{\imath} + (A_y - B_y)\hat{\jmath} \tag{4.14}$$

$$c\vec{A} = (cA_x)\hat{\imath} + (cA_y)\hat{\jmath} \tag{4.15}$$

$$\frac{d\vec{A}}{dt} = \frac{dA_x}{dt}\hat{\imath} + \frac{dA_y}{dt}\hat{\jmath} \tag{4.16}$$

$$\vec{A}\cdot\vec{B} = A_xB_x + A_yB_y \tag{4.17}$$

Note that all these rules, except the last one, can be called "component-wise" rules. I.e., identity, addition, subtraction, scaling, and differentiation operations of vectors can be performed component-wise. This is the basis of "**divide and conquer** rule" that we will use in the next lecture. Namely, a 2D problem can be divided into 2 sets of easy 1D problems, using the component representation.

What more about vectors?

This lecture summarized all things about vectors, to the extent as necessary in this course, except one thing – the vector product, $\vec{A} \times \vec{B}$. We will cover that when we learn about torque and angular momentum, later in this course. The vector product is an odd operation – it is not commutative, but rather anti-commutative: $\vec{B} \times \vec{A} = -\vec{A} \times \vec{B}$. In contrast, the scalar product is a commutative operation, as already mentioned above: $\vec{B} \cdot \vec{A} = \vec{A} \cdot \vec{B}$.