GRAVITY

Chap. 8 of Wolfson

What you will learn

- Kepler's laws
- **Newton's law of gravitation**
- Gravitational force field
- **Newton's "shell theorem"**
- Different orbits in inverse square law force field
- **Circular motion under gravity**
- **Escape speed, gravitational potential energy**

Kepler's superb insights …

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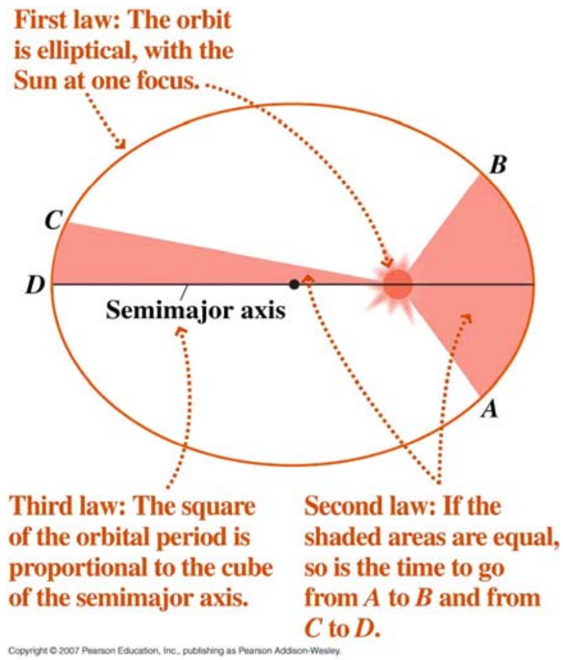

Newton's law of gravitation

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$$F = \frac{Gm_1 m_2}{r^2}$$

- Two masses attract each other
- An "inverse square law"

  $F \propto r^{-2}$ where $r =$ distance between two masses
- Not only explained motions of planets, but also laid basis to other subsequent developments (Coulomb force between electric charges, a subject of 6C).
- Constant of universal gravitation $G = 6.67 \times 10^{-11}\ \mathrm{Nm^2/kg^2}$
- The origin of the gravitational acceleration *g* ("small g")
  - Close to the earth surface,

    $F \approx mg, \quad g = \frac{GM_E}{R_E^2} = 9.81\ \mathrm{m/s^2}$ $(M_E = 5.97 \times 10^{24}\ \mathrm{kg},\ R_E = 6.37 \times 10^6\ \mathrm{m})$
  - At height *h* from the surface, $F \approx ma, \quad a = \frac{GM_E}{(R_E+h)^2}$

(cf. Ex. 8.1)

Quiz

- Suppose the distance between two objects is cut in half. The gravitational force between them is …

  A. doubled.

  B. halved.

  C. unchanged.

  D. quadrupled.

  E. quartered.

Force field

- Consider an object with mass *M* (e.g. the earth). A small "test mass" *m* will experience a force

$$\vec{F} = -\frac{GMm}{r^2}\hat{r} = m\vec{g}(\vec{r})$$

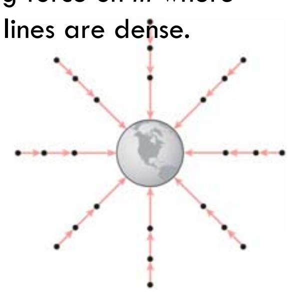
$$\vec{g}(\vec{r}) = -\frac{GM}{r^2}\hat{r}$$

$\hat{r}$ is a unit vector, pointing outward radially.
So, there is an overall minus sign for $\vec{F}, \vec{g}$.

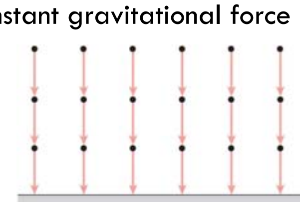
- Here, $\vec{g}(\vec{r})$ is called "force field."

$\vec{g}(\vec{r}) \approx -g\hat{j}$ where $g = 9.8 \text{ m/s}^2$,
independently of the position vector $\vec{r}$,
only near the surface of the earth.

Non-uniform field vectors/lines.
Strong force on *m* where field lines are dense.



Uniform field vectors/lines.
Constant gravitational force on *m*.



Force field – what's the fuss?

This slide is for optional reading.

- Did we really introduce anything "real" or "substantial" than just a mathematically trivial "invention" – a "field"?!

- Answer: YES! In a more complete/advanced view, the "field" in this case has the physical significance as "something exchanged between two masses." So, instead of two masses interacting with each other directly at a distance (the "action at a distance non-sense," which has bothered the best minds), they interact by exchanging "field particles." So, in this *microscopic* sense, all forces are contact forces! For the electromagnetic force (to be covered 6C), the field particles are photons, quanta of light. One caveat: for the gravitational force, the (surely Nobel-worth) detection of field particles has not been realized yet.

Newton's shell theorem

Proof will be given in Phys 6C.

- Symmetry, symmetry, spherical symmetry

Consider a body with a mass density distribution that is dependent only on r (i.e., is spherically symmetric). A uniform mass density is a simple such example.
$dM \equiv$ mass of a spherical shell with radius r and thickness dr.
Define function $\rho(r)$ as $dM = \rho(r)dr$.
Put a small test mass m at position vector $\vec{r}$.
Newton's shell theorem: The force it experiences is
$\vec{F} = -G\frac{mM(r)}{r^2}\hat{r}$ **where** $M(r) = \int_0^r \rho(r')dr'$.

- In other words, (i) forget about stuff at radius > r, (ii) treat the total mass of all stuff at radius < r as a point mass at the origin.
- A similar theorem holds for Coulomb force (Phys 6C).

Quiz – Journey through the center of the earth (just for fun!)

- Let us assume (just for fun) that someone succeeded in building a straight duct through the center of the earth. Suppose you drop into that duct. What will happen to you (assuming no air resistance or heat is involved)?
  - A. Oscillate back and forth
  - B. Come out through the other side and disappear into space

Journey through the center of the earth – just for fun!

- Use Newton's shell theorem!
- Simple Harmonic Motion, assuming a uniform density!

$$\vec{F} = -G\frac{mM(r)}{r^2}\hat{r}$$

$$M(r) = M_E\frac{r^3}{R_E^3}$$

$$\vec{F} = -G\frac{mM_E}{R_E^3}r\hat{r}$$

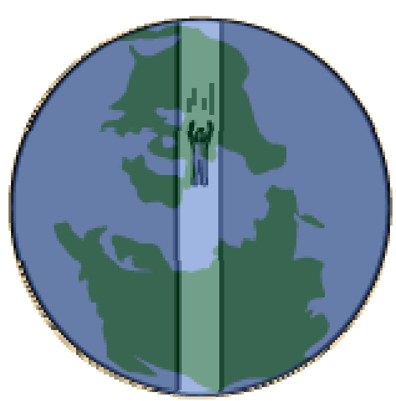
Hookes law! Simple Harmonic Motion!

$$k = \frac{GmM_E}{R_E^3}$$

$$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{GM_E}{R_E^3}}$$

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{R_E^3}{GM_E}}$$

$T = 84$ minutes (round trip!)



http://hyperphysics.phy-astr.gsu.edu

Motions in gravitational field – Newton's bold physical insight

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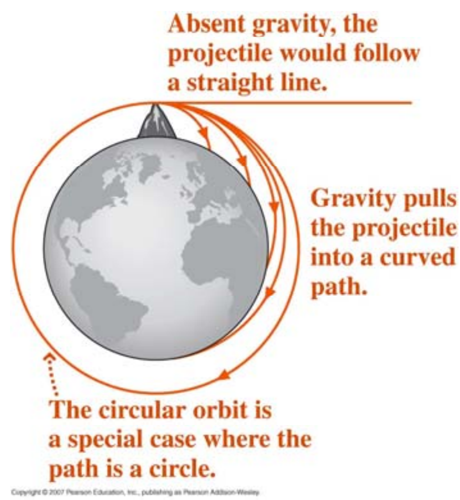
- What if apple here and the moon up there are governed by the same law?! (concept of symmetry)



Motions in gravitational field – "Falling" in different styles

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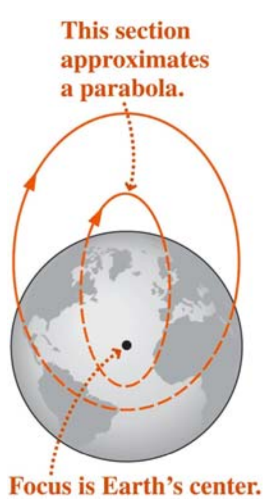
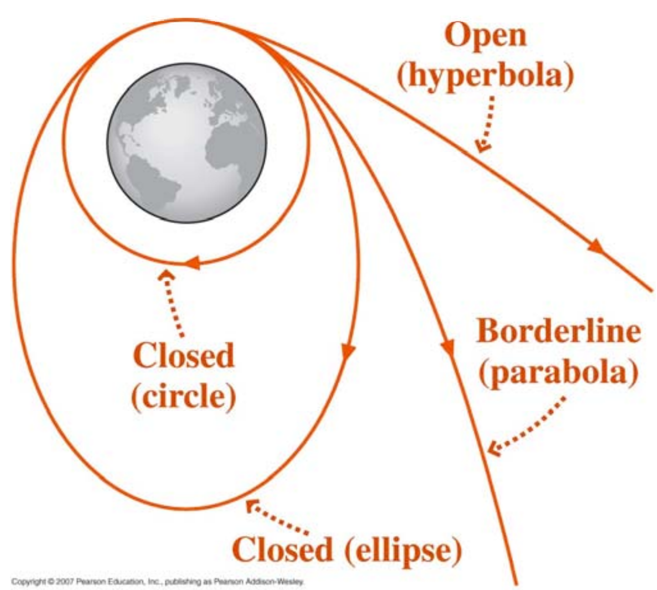
Elliptical orbits (closed) (interrupted)





Circular motion under gravity Example – geosynchronous orbit

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- Geosynchronous orbit: circular orbit with T = 1 day
- Altitude required for geosynchronous orbit?

Circular motion at an altitude h!
centripetal accel. (a_r) = inward radial force / mass (F_r/m)
distance from the center of earth $= R + h$.
$a_r = \omega^2/(R+h)$, $\omega = 2\pi/T$.
$F_r/m = GM_E/(R+h)^2$ from Newton's law of gravitation.
$\therefore (R+h)^3 = \frac{GM_E}{\omega^2} = \frac{GM_E T^2}{4\pi^2}$

(We just proved Keplers 3rd law for circular orbits!)

Solving for h, $h = (\frac{GM_E T^2}{4\pi^2})^{1/3} - R_E$.
Using $T = 1$ day (geosynchronous orbit), we get $h = 3.6 \times 10^4$ km.

Gravitational Potential Energy

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$\vec{F} = -\frac{GMm}{r^2}\hat{r}$ for mass m in the field due to M.
Only r is involved (spherical symmetry).
Effectively a one-dimensional problem.
What is $U(r)$ such that $\vec{F} = -\frac{dU}{dr}\hat{r}$?

Ans: $U(r) = -\frac{GMm}{r}$ $\quad \because \frac{d}{dr}(1/r) = -1/r^2$
$U(r)$ = gravitational potential energy

Mechanical Energy Conservation
$E = K + U = \frac{1}{2}mv^2 - \frac{GMm}{r}$ = constant.
(independent of time, and thus position)

Escape Speed

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- An object shot up may not come back down! What is such an escape condition?

Mechanical Energy Conservation
$E = \frac{1}{2}mv^2 - \frac{GMm}{r} = \text{constant (independent of } r\text{)}.$

Condition: $E \geq 0 \quad \because E = E(r \to \infty) = \frac{1}{2}mv^2_{r\to\infty} \geq 0.$
v_{esc} is the minimum speed necessary. $\frac{1}{2}mv^2_{esc} = \frac{GMm}{r}$
$v_{esc} = \sqrt{\frac{2GM}{r}}$
At earth surface, $v_{esc} = 11.2$ km/s.

Circular motion and potential energy

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Mechanical Energy Conservation
$E = K + U = \frac{1}{2}mv^2 - \frac{GMm}{r} = \text{constant.}$

In a circular motion, r is constant by definition!
Which means v is constant, since E is constant.
The centripetal force equation: $mv^2/r = GMm/r^2$
Which means $mv^2 = GMm/r$, or $2K = -U$!

Thus, for a circular motion under gravity:
$E = K + U = -K = \frac{1}{2}U < 0$

Quiz – Circular Motion

- Suppose Spacecraft A and Spacecraft B are in circular orbits around the Earth, with Spacecraft B at a higher altitude. Which one of the following statements is true?
  - A. Spacecraft B moves faster.
  - B. Spacecraft B has greater total energy *E*.
  - C. A larger proportion of Spacecraft B's total energy is kinetic.
  - D. Spacecraft A has greater potential energy *U*.
  - E. None of the above statements are true.

Concluding with ? … !

- Can you derive g from Newton's law of gravitation?
- The (wildly fictional) journey through the center of the earth would be a SHM. How so?
- For a circular motion in a gravitational field,
  - What is the relationship between r and T ("geosynchronous orbit" or Kepler's third law problem)?
  - What is the relationship between K, U, and E?
- What goes up sometimes does not come down. How so?