LECTURE XXI

Simple Harmonic Motion

This lecture will deal mainly with the simple harmonic motion (SHM), whose importance in physics cannot be over-emphasized. It is also a quite sweet-sounding subject – it is *simple* and *harmonic*! Indeed, the SHM is simple (to understand), pleasant (e.g., it gives us music and light), and very profound (e.g., some current theory suggests that *all* things in nature are derived from SHMs of one form or another).

The SHM is also very tightly connected with the circular motion, that we just studied in some gory details. You will see why in the following.

21.1. DEFINITION. **Simple Harmonic Motion, Angular Frequency, and Amplitude**

Consider a generalized "position variable" q, which fully describes the motion of a point object. The meaning of q depends on each problem. For example, q corresponds to the real position x for a linear motion, or to the angle θ for a motion on a circular path. In any case, if the motion is described by the formula

$$q = A\cos(\omega t + \phi) \tag{xxi.1}$$

where A, ω, ϕ are time independent constants, then the motion is called a simple harmonic motion (SHM). In this context, ω is called the ***angular frequency***, A is called the ***amplitude*, and** ϕ is called the ***initial phase***, or often just the ***phase [constant]***.

21.2. FACT. *Consider an **imaginary** uniform circular motion in a plane, whose coordinates are given as (q, s). A simple harmonic oscillation in q corresponds to this **imaginary** uniform circular motion with **the radius A, the angular speed ω, and the initial angle ϕ**.*

PROOF. Recall from previous work that a uniform circular motion with the angular speed ω and the radius R is described by the position vector $\vec{r} = (x, y)$ with $x = R\cos(\omega t)$ and $y = R\sin(\omega t)$. But this was a particular motion, whose initial condition was such that it starts at $x = R$ and $y = 0$. That is, the initial angle $\theta_0 \equiv \theta(t = 0) = 0$ for this particular motion. The initial angle θ_0 certainly does not have to be zero for a uniform circular motion. For a general nonzero θ_0, we can write $\theta = \theta_0 + \omega t$, and so $x = R\cos\theta = R\cos(\omega t + \theta_0)$ and $y = R\sin\theta = R\sin(\omega t + \theta_0)$. The proof is complete, if we rename $x \to q$, $y \to s$, $R \to A$ and $\theta_0 \to \phi$. □

21.3. NOTE. **All names are "dummy"**

Often names can be confusing, if we re-use them for different things. Sorry about that, but often the situation is such that there is no easy way to fix this problem. We just witnessed one such example. So, please be alert, and keep in mind that names are just those. If you look at Definition 21.1, I mention that q can mean x or θ depending on what actual motion we consider. Then in the proof of Fact 21.2, I use x, θ, θ_0 etc., only to rename them some other things in the end. Please be alert not to mix them up! x and θ in Definition 21.1 are different from x and θ in the proof of Fact 21.2.

21.4. OBSERVATION. ***Imaginary Circles and Reality*** **(optional)**

Please note again that the circle described in Fact 21.2 is *imaginary*. That is, the circle does *not* exist in reality, in general. We will see a few examples of the SHM below, and, except for one case, the circular motion is only in our imagination.

In the case of a SHM, one can say that the reality (q) is a projection of what is going on, i.e. a uniform circular motion, in space (q, s) that is partly "imaginary" since s is an imaginary axis that we pulled out of our hat.

Mathematically, this correspondence has a lot to do with the Euler relation, $e^{i\theta} = \cos\theta + i\sin\theta$, discussed in Appendix B. Indeed, any complex number $z = x + iy$ can be written also as $z = re^{i\theta}$, where $r \equiv |z|$. Using the Euler's relation, we see that $x = r\cos\theta$ and $y = r\sin\theta$, just as for a two dimensional vector $\vec{r} = (x, y)$ in the (r, θ) notation where $r \equiv |\vec{r}|$ (which like $|z|$ is given by $\sqrt{x^2 + y^2}$, of course). For this reason, a two dimensional vector space (i.e. a plane) is often (when relevant) referred to as the *complex plane*, whose y axis is then called the imaginary axis. In the fantastic sub-field of mathematics of complex numbers ("complex analysis"), the reality (i.e. real numbers) is merely a projection of a fuller space, i.e. the complex plane! Complex analysis comes handy for all sorts of things, e.g. AC circuit analysis, which is basically a study of SHMs in the electricity.

21.5. DEFINITION. **Periodic motion (not just SHM), frequency f, and period T for a periodic motion**

Suppose a motion is described by a position vector $\vec{r}$ and a velocity vector $\vec{v}$, and that $\vec{r}(t + S) = \vec{r}(t)$ and $\vec{v}(t + S) = \vec{v}(t)$, for *any* t and some non-zero constant S. Then, this motion is called periodic. Consider all possible positive S values, and the minimum value of those S values. That minimum value is called the period and is usually denoted as T. Given T, the frequency is given as

$$f \equiv \frac{1}{T}$$

f has[1] the unit of 1/sec, or equivalently Hz. Note that f and T are defined for any periodic motion, not just a SHM. For a SHM, the relation between the frequency (f) and the angular frequency (ω) is

$$\omega = 2\pi f$$

21.6. EXAMPLE. **Mass on a spring and solving Newton's equation**

Consider a mass attached to one end of a spring that can move horizontally[2]. The other end of the spring is fixed. The potential energy $U(x) = \frac{1}{2}kx^2$ and the force is $F(x) = -kx$ (Hooke's law). Newton's equation reads

$$m\ddot{x} = -kx \qquad \text{(xxi.2)}$$

Observe that, since the acceleration is *not* constant, but a function of $x(t)$, which we want to solve for, this problem is qualitatively different from the "projectile motion" motion. At first sight, it appears that the problem is hopelessly difficult. Why? Well, we like to know what is $x(t)$, but both the LHS and the RHS contain $x(t)$ and so it looks like there is no way to solve this type of equation. This is actually far from

[1] Another symbol commonly used for frequency is ν, but unfortunately the textbook uses ν to mean v (velocity), so we will not use ν to mean frequency in this course.

[2] The treatment in the case of the vertical motion yields the same result as below, while in that case there is the gravitational force changes the equilibrium length of the spring in comparison to that in the horizontal geometry.

being true, and it is trivial to show that this type of "differential equation" is solvable and it is also equally trivial to actually solve it using a computer.

To see why I will explain using Eq. xxi.2 as an example, although the argument holds much more generally. First, suppose one knows the initial conditions, $x_0 \equiv \ddot{x}(t=0)$ and $v_0 \equiv \dot{x}(t=0)$, at $t=0$. Second, note that this means that we know the RHS of Eq. xxi.2 at $t=0$, *and thus* that we know the LHS at $t=0$. Namely, we know $\ddot{x}(t=0) \equiv a_0$. Third, now we know, at least at $t=0$, the position, the velocity, and the acceleration. This is a very good thing. Why? Suppose we take an infinitesimal time interval dt and then ask the question "do we have anything to say about x, v, and a at $t = 0 + dt$?". The answer is "yes, we can!". Since dt is infinitesimal, we have no problem in assuming that a is constant *during the time interval* [0, dt], and indeed we do know how to do problems in the constant a case! More explicitly, $x(dt) = x_0 + v_0 dt$ and $v(dt) = v_0 + a_0 dt$. Well, this is the essence of proving that differential equations are solvable. In order to know $x(t)$ at any time t, all we need to do is to repeat the procedure N times where $N = t/dt$.

This is precisely how a computer solves a differential equation like Eq. xxi.2. In that case, we say that the differential equation is solved "numerically." In a few rare cases, a differential equation can be also solved with pencil and paper, or more formally speaking "analytically." In either case, note that there is one thing to keep in mind. To solve Newton's equation for a variable x, we need two numbers for the initial conditions, x_0 and v_0. This has to do with the fact that the highest derivative in Newton's equation is the 2nd derivative, or in other words, the fact that Newton's equation is a 2nd order differential equation.

Having obtained a general understanding of Newton's equation, we will now demonstrate that the solution to Eq. xxi.2 is a SHM, i.e. the solution is in the form of Eq. xxi.1 with $q = x$ and thus prove that Eq. xxi.1 provides the most general solution to Eq. xxi.2. At this point, you might say that this is a bit of cheating since we present the solution first! This is true. However, since the mathematics of analytically solving differential equations is outside the scope of this course, our approach is a reasonable one, and is actually not so shabby compared to the in-principle method of obtaining the solution.

So much for the background, Eq. xxi.1 with $q = x$ reads

$$x = A\cos(\omega t + \phi)$$

which, when differentiated twice, gives

$$\ddot{x} = -\omega^2 x$$

which means

$$m\ddot{x} = -m\omega^2 x$$

Since this has to be equal to $-kx$, we see that if

$$\omega = \sqrt{\frac{k}{m}} \tag{xxi.3}$$

then $x(t)$ is the solution for Newton's equation, Eq. xxi.2, for any values of A and ϕ. So, ω is not an abitrary number. Rather, it is determined uniquely by the physical parameters of the problem, the mass and the spring constant. In this sense, ω is called the ***natural [angular] frequency*** of the system. On the other hand, A and ϕ are arbitrary constants. Why do we have two of them? It is precisely because the initial conditions are specified by two constants x_0 and v_0. Indeed, one can see that $x_0 = A\cos\phi$ and $v_0 = -A\omega\sin\phi$. Accordingly, specifying two constants x_0 and v_0 is equivalent to specifying two constants A and ϕ.

21.7. EXAMPLE. **A simple pendulum and a physical pendulum, with a small amplitude**

A **simple pendulum** consists of a point mass attached to a massless string, which hangs down from a pivot point. It is a simplified model of "a child on a swing" or "Tarzan on a vine" in problems that you have done.

So, consider a simple pendulum with mass m and length L. Since the motion is a purely rotational one, we can use the rotational form of Newton's equation $\tau = I\alpha$ (Theorem 19.1), relative to the pivot point. $\tau = -F_g L \sin\theta$ (minus sign means that the torque decelerates the angular motion; no torque due to tenstion since tension is parallel to the position vector of the mass) where $F_g = mg$ and θ is the angle between the vertical axis and the string. $I = mL^2$ and $\alpha = \ddot{\theta}$. Summarizing, Newton's equation reads

$$-mgl\sin\theta = mL^2\ddot{\theta}$$

which simplifies to

$$-g\sin\theta = L\ddot{\theta}$$

This equation cannot be solved analytically, and so in general the solution should be obtained numerically. However, *if the amplitude of the swinging motion is small* such that θ remains small (much smaller than 1 in radian), then one can approximate $\sin\theta \approx \theta$ (Appendix B), and the above equation becomes

$$L\ddot{\theta} = -g\theta$$

Note the similarity of this equation to Eq. xxi.2, if one makes the mapping $x \to \theta$, $m \to L$ and $k \to g$. So, a simple pendulum with a small amplitude is a SHM, with

$$\omega = \sqrt{\frac{g}{L}}$$

A **physical pendulum** is an arbitrarily shaped object swinging with respect to a pivot point. In that case, the only difference from a simple pendulum is that (1) L should be interpreted as the distance from the pivot point to the *center of mass* of the object, and (2) I is in general not mL^2. On the other hand, on dimensional grounds, $I = AmL^2$ where A is a dimensionless number that is dependent only on the shape of the object. The natural frequency is then given by $\omega = \sqrt{g/(AL)}$.

21.8. EXAMPLE. **The torsion oscillator**

When a wire is twisted slightly by a small angle (θ) around its long axis, it tends to like to come back to the untwisted state. This gives rise to a restoring torque, $\tau = -\kappa\theta$, which is the rotational analogue of Hooke's law $F_{spring} = -kx$. If an object with the rotational inertia I is attached at the end of the wire, then we have

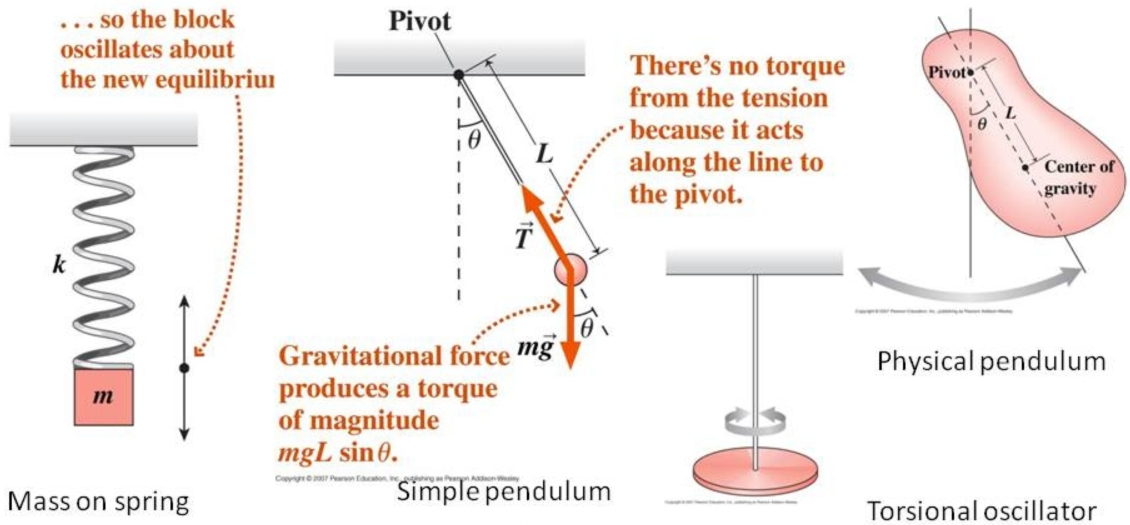
$$\tau = I\alpha = I\ddot{\theta}$$

namely

$$I\ddot{\theta} = -\kappa\theta$$

which is another form of a SHM equation, for which the natural frequency is given by, in analogy with the above examples,

$$\omega = \sqrt{\frac{\kappa}{I}}$$



Various Simple Harmonic Osillators

21.9. NOTE. Energy of a SHM

Consider the mass on spring problem with the amplitude A. When $x = \pm A$, the kinetic energy is zero, and the total mechanical energy is $E = \frac{1}{2}kA^2$. When $x = 0$, the elastic energy is zero, and the speed attains its maximum v_{max} and the total mechanical energy is $E = \frac{1}{2}mv_{max}^2$. Of course, the total energy is conserved if the spring force is the only force involved.

$$
\begin{aligned}
E &= \frac{1}{2}kA^2 \text{ (at end points)} \\
&= \frac{1}{2}mv_{max}^2 \text{ (at the center/zero point)} \\
&= \frac{1}{2}kx^2 + \frac{1}{2}mv^2 \text{ (at any point)}
\end{aligned}
$$

Namely, there is a continual conversion from the potential energy to the kinetic energy and vice versa, during the periodic motion, while the total mechanical energy is always conserved.