Angular Momentum and Its Conservation

20.1. DEFINITION. **Angular Velocity (resivited)**

We define the angular velocity vector $\vec{\omega}$ using the right-hand rule. Namely, for a rotation, $\vec{\omega}$ points along the rotational axis, according to the rule of the following figure.

20.2. DEFINITION. **Angular Acceleration (revisited)**

$$\vec{\alpha} \equiv \frac{d\vec{\omega}}{dt}$$

20.3. DEFINITION. **Cross product of two vectors**

For given vectors $\vec{A}$ and $\vec{B}$, which make an angle θ between each other, the cross product denoted as

$$\vec{A} \times \vec{B}$$

is defined by the magnitude

$$\left|\vec{A} \times \vec{B}\right| = AB\sin\theta$$

and the direction, which is determined by the right hand rule, where the sense of rotation is defined as the rotation of $\vec{A}$ towards $\vec{B}$, with $\vec{A}$ and $\vec{B}$ sharing the common starting point, and the "thumb direction" gives the direction of $\vec{A} \times \vec{B}$.

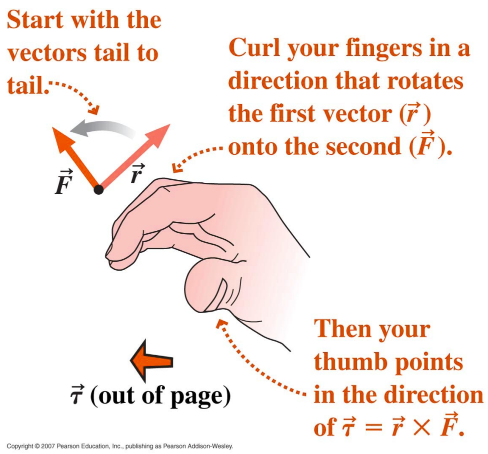
20.4. FACT. *Some basic properties of the cross product*

(1) $\vec{A} \times \vec{A} = 0$, *(2)* $\vec{A} \times \vec{B} = -\vec{B} \times \vec{A}$, *(3)* $\vec{A} \times (\vec{B} + \vec{C}) = \vec{A} \times \vec{B} + \vec{A} \times \vec{C}$

PROOF. (1) Because $\theta = 0$. (2) From the right hand rule. (3) This is a bit complicated to prove at this point. It follows from the fact that $\vec{A} \times \vec{B} = (A_yB_z - A_zB_y, A_zB_x - A_xB_z, A_xB_y - A_yB_x)$, which can be shown to be equivalent to the above definition of the cross product. ☐

20.5. DEFINITION. **Torque vector $\vec{\tau}$ (revisited)**

$$\vec{\tau} \equiv \vec{r} \times \vec{F}$$



Start with the vectors tail to tail.

Curl your fingers in a direction that rotates the first vector ($\vec{r}$) onto the second ($\vec{F}$).

Then your thumb points in the direction of $\vec{\tau} = \vec{r} \times \vec{F}$.

20.6. DEFINITION. **Angular momentum**
is defined as

$$\vec{L} \equiv \vec{r} \times \vec{p}$$

where $\vec{p}$ is the linear momentum ($m\vec{v}$). For a compound object, $\vec{L} = \sum_i \vec{r}_i \times \vec{p}_i$, or (equivalently) $\int \vec{r} \times \vec{v} dm$.

20.7. THEOREM. ***Newton's 2nd Law in terms of $\vec{L}$ and $\vec{\tau}$.***

$$\frac{d}{dt}\vec{L} = \vec{\tau}_{tot} \tag{xx.1}$$

If internal torques cancel (see Theorem 19.1 for the condition that this will happen), which we will assume to be the case in this course, then

$$\frac{d}{dt}\vec{L} = \vec{\tau}_{net} \tag{xx.2}$$

where $\vec{\tau}_{net}$ is the sum of torques to due to ***external*** *forces.*

PROOF. $\vec{L} = \sum_i \vec{r}_i \times \vec{p}_i$. When applying $\frac{d}{dt}$, note that the product rule applies.

$$\frac{d}{dt}\vec{L} = \sum_i \dot{\vec{r}}_i \times \vec{p}_i + \sum_i \vec{r}_i \times \dot{\vec{p}}_i$$

The first term is 0, since $\vec{p}_i = m_i \dot{\vec{r}}_i$ is parallel to $\dot{\vec{r}}_i$ (thus $\theta = 0$). Noting that $\dot{\vec{p}}_i = \vec{f}_i$ (*the total* force acting on the particle i) we have

$$\frac{d}{dt}\vec{L} = \sum_i \vec{r}_i \times \vec{f}_i = \sum_i \vec{\tau}_i = \vec{\tau}_{tot}$$

In normal cases in which internal torques cancel, $\vec{\tau}_{tot}$ can be replaced by $\vec{\tau}_{net}$, the sum of all torques due to *external* forces. ☐

NOTE. **Warning:** $\tau_{net} = I\alpha$ **and** $\vec{\tau}_{net} = \frac{d}{dt}\vec{L}$. **(optional advanced topic, but please keep in mind the last summarizing paragraph at the end!)** The latter is more fundamental than the former, which was derived in Theorem 19.1. There are a few reaons. (1) Note that both $\vec{\tau}$ and $\vec{L}$ are *dependent* on the choice of the origin of the coordinate system, while I

and $\alpha = |\dot{\vec{\omega}}|$ are dependent only on the choice of the rotational axis only but *not* on the choice of the origin. So, obviously $\tau_{net} = I\alpha$ is not a generally valid equation. In fact, it is valid only when the origin is chosen in a "highly symmetric manner." Most choices that we make in this course and most choices that people make fall into this category, so one does not need worry too much about this, but nevertheless it may be good to be aware that $\tau_{net} = I\alpha$ is not valid if you put the origin of the coordinate system at a very weird place. However, $\underline{\vec{\tau}_{net} = \frac{d}{dt}\vec{L}}$ remains valid for *any* inertial reference frame, regardless of where the origin is, or for a center of mass reference frame. (2) $\underline{\vec{\tau}_{net} = \frac{d}{dt}\vec{L}}$ remains valid even if the rotational inertia is *not* constant. (3) **You might also wonder whether $\vec{L} = I\vec{\omega}$ is true**. This is **generally *not* true**, but is true if the coordinate system is taken in a "highly symmetric manner" and if the object under consideration is "highly symmetric." Fortunately, all cases we deal with in this course fall into this category.

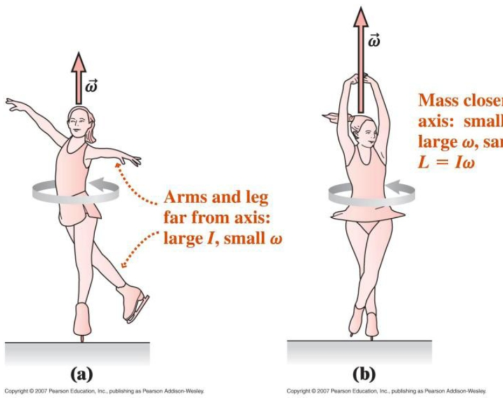
All in all, you will need to remember that $\underline{\vec{\tau}_{net} = \frac{d}{dt}\vec{L}}$ is a very robust equation, but $\tau_{net} = I\alpha$ and $\vec{L} = I\vec{\omega}$ are *not*. With this caution, you can however take comfort in the fact that, in *this* course, we will consider only those highly symmetric problems, for which $\vec{L} = I\vec{\omega}$ is always valid and for which $\tau_{net} = I\alpha$ is valid if the problem is a *quantitative* one.

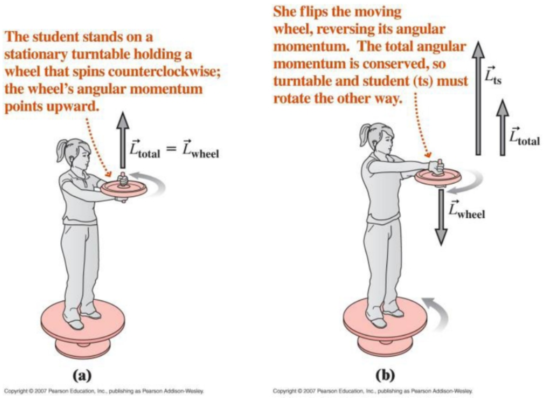
20.8. THEOREM. ***Angular Momentum Conservation***

In the absence of external torques, e.g. in an isolated system, the angular momentum is conserved.

$$\frac{d}{dt}\vec{L} = 0$$

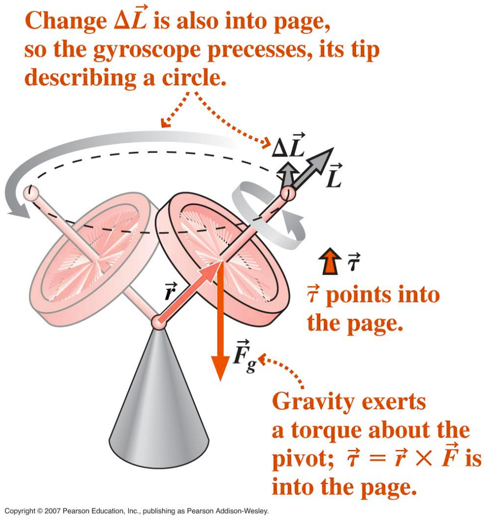
PROOF. A direct consequence of Eq. xx.2. □





20.9. FACT. ***Fast rotating objects – Gyroscopes, tops, etc.***

Fast rotating objects behave in a non-intuitive way (e.g. they seem to defy gravity). However, considering the angular momentum and the torque, a general understanding can be gained. For instance, a precession is a consequence of the small torque applied to a fast spinning top or gyroscope.


Change $\Delta\vec{L}$ is also into page, so the gyroscope precesses, its tip describing a circle.
$\Delta\vec{L}$
$\vec{L}$
$\vec{\tau}$
$\vec{\tau}$ points into the page.
$\vec{r}$
$\vec{F}_g$
Gravity exerts a torque about the pivot; $\vec{\tau} = \vec{r} \times \vec{F}$ is into the page.