

LECTURE XIX

Rotational Motion (cont.)

19.1. THEOREM. *Newton's 2nd Law for a Rotational Motion*

Suppose that an object is undergoing a pure rotational motion, in an **inertial reference frame** or a **center of mass reference frame** (i.e. a reference frame whose origin is the center of mass of the object under consideration), then

$$\tau_{\text{tot}} = I\alpha \quad (\text{xix.1})$$

where α is the angular acceleration, and τ_{tot} is the total torque, i.e. sum of all torques applied to the constituent particles of the object.

If internal forces between two internal particles are directed along the line connecting the two particles, internal torques cancel each other, and τ_{tot} is reduced to the net external torque, τ_{net} . This case is common and will be assumed to be valid in this course. In this important case, we have

$$\tau_{\text{net}} = I\alpha \quad (\text{xix.2})$$

where τ_{net} is sum of torques due to **external** forces.

PROOF. The content of this theorem, while useful to know, is advanced in the sense that its full proof is not offered at this level of physics courses. It is not difficult, but involves a lengthy vector algebra. So, here goes a partial proof of this theorem, only for the most basic part. [If you are curious about the full proof, you are welcome to ask me.] By definition, everything is rotating at ω , so define $L \equiv \sum_i m_i r_i^2 \omega = I\omega$. $dL/dt = \sum_i m_i r_i^2 \alpha = I\alpha$. On the other hand, $\sum_i m_i r_i^2 \alpha = \sum_i m_i r_i a_{i,t}$ (from Eq. xvii.1) $= \sum_i r_i f_{i,t} = \sum_i \tau_i$ (from Eq. xvii.3; f_i is the total force acting on particle i and $f_{i,t}$ is the tangential component of it) $= \tau_{\text{tot}}$. \square

The equation of motion that we just proved is a "translation" of the Newton's 2nd law in one dimension, $F = ma$, using the "translation table" $F \rightarrow \tau$, $m \rightarrow I$ and $a \rightarrow \alpha$ (and $x \rightarrow \theta$, $v \rightarrow \omega$). Also, note that a center of mass frame is usually an accelerated frame, which is *not* an inertial reference frame. Nevertheless, it is a special reference frame in which the above law is valid.

19.2. THEOREM. *Rotational Kinetic Energy*

For an object in a pure rotation, the kinetic energy is given by $K_{\text{rot}} = \frac{1}{2}I\omega^2$

PROOF. $K = \sum_i \frac{1}{2}m_i v_i^2$, while $v_i = D_i \omega$, where D_i is the distance between particle i and the rotational axis. $K = \frac{1}{2}\omega^2 \sum_i m_i D_i^2$. Thus, $K = \frac{1}{2}I\omega^2$, since $I = \sum_i m_i D_i^2$. \square

19.3. THEOREM. Work-Energy Theorem for Pure Rotation

For an object in a pure rotation, the net work done on the object from an initial time/state (i) to the final time/state (f) is given by $W(i \rightarrow f) = \frac{1}{2}I\omega_f^2 - \frac{1}{2}I\omega_i^2$, where $W(i \rightarrow f) = \int_{\theta_i}^{\theta_f} \tau d\theta$.

PROOF. In a previous lecture, we proved that $W(i \rightarrow f) = K_f - K_i$, where K is the kinetic energy. For a pure rotation, we just proved that $K = \frac{1}{2}I\omega^2$. Lastly, to prove $W(i \rightarrow f) = \int_{\theta_i}^{\theta_f} \tau d\theta$, note that $W(i \rightarrow f) = \int F_t dl$ where F_t is the tangential component of the net force and dl is the infinitesimal displacement of a circular motion. We know that $dl = r d\theta$ and $F_t r = \tau$, so this completes the proof. \square

19.4. THEOREM. **Total kinetic energy for a general compound object**

$K_{total} = K_{cm} + K_{internal}$ where $K_{cm} = \frac{1}{2}MV_{cm}^2$ and $K_{internal} = \frac{1}{2}\sum_i m_i v_{i,cm}^2$ where $V_{cm} = |\dot{\vec{R}}_{cm}|$ and $v_{i,cm} = |\dot{\vec{r}}_{i,cm}|$ where \vec{R}_{cm} is the center of mass coordinate, and $\vec{r}_{i,cm}$ is the position vector of particle i relative to the center of mass \vec{R}_{cm} (i.e. $\vec{r}_{i,cm}$ is the position in the center of mass reference frame).

PROOF. $\vec{r}_i = \vec{R}_{cm} + \vec{r}_{i,cm}$. The total kinetic energy is given by

$$\begin{aligned} K_{tot} &= \frac{1}{2} \sum_i m_i \dot{\vec{r}}_i \cdot \dot{\vec{r}}_i \\ &= \frac{1}{2} \sum_i m_i (\dot{\vec{R}}_{cm} + \dot{\vec{r}}_{i,cm}) \cdot (\dot{\vec{R}}_{cm} + \dot{\vec{r}}_{i,cm}) \\ &= K_{cm} + K_{internal} + \dot{\vec{R}}_{cm} \cdot \sum_i m_i \dot{\vec{r}}_{i,cm} \end{aligned}$$

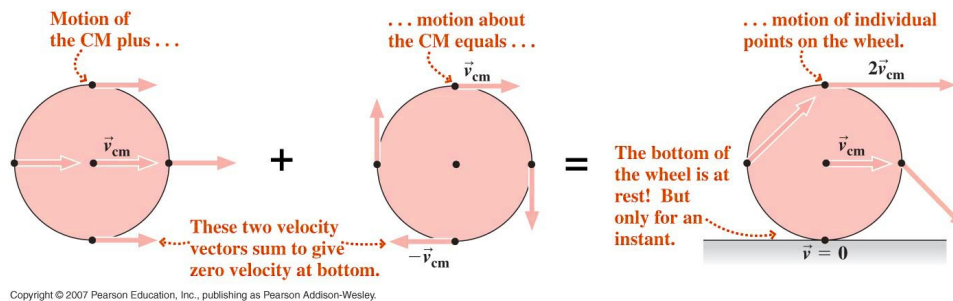
Note that by definition, $\sum_i m_i \vec{r}_{i,cm} = 0$ and thus $\sum_i m_i \dot{\vec{r}}_{i,cm} = 0$, and so the last term vanishes. \square

19.5. THEOREM. **Total kinetic energy for a rolling wheel**

For a rolling wheel (with uniform density), $K_{total} = K_{cm} + \frac{1}{2}I_{cm}\omega^2$ where $K_{cm} = \frac{1}{2}MV_{cm}^2$ as given above and I_{cm} is the rotational inertia around the center of mass.

PROOF. For a rolling wheel, the motion relative to the center of mass is a pure rotation. So, $K_{internal} = \frac{1}{2}I_{cm}\omega^2$ due to Theorem 19.2. \square

In connection with this theorem, it is essential to understand the following diagram.



19.6. EXAMPLE. The wheel is a great invention of the human kind. Let us ask the following questions about it. (1) A wheel is rolling down an incline without slipping. Is there a frictional force acting on it? (2) A wheel is rolling, without deformation, on a horizontal surface. Is there a frictional force acting on it? (3) In both cases (1) and (2), is the mechanical energy conserved? Ans: Yes, No, Yes.