LECTURE XVIII

Rotational Motion (cont.)

18.1. NOTE. D_i in terms of $\vec{r}_i$ and $\hat{\omega}$ (optional)

Note that, different from the textbook, I use the notation D_i or D instead of r_i or r. This is so that D_i or D is not misunderstood as $|\vec{r}_i|$ or $|\vec{r}|$. If not, then, what is it? Here is the answer, *assuming that we use the coordinate system whose origin lies on the rotation axis*.

$$D_i^2 = [\vec{r}_i - (\vec{r}_i \cdot \hat{\omega})\hat{\omega}] \cdot [\vec{r}_i - (\vec{r}_i \cdot \hat{\omega})\hat{\omega}] \tag{xviii.1}$$

where $\hat{\omega}$ **is defined as the unit vector along the rotation axis**. In plain words, the meaning of this formula is the following. Project $\vec{r}_i$ along the rotation axis. That projection is $(\vec{r}_i \cdot \hat{\omega})\hat{\omega} \equiv \vec{r}_{i,\|}$. Subtract this from the original vector, then you get $\vec{r}_i - (\vec{r}_i \cdot \hat{\omega})\hat{\omega} \equiv \vec{r}_{i,\perp}$. Note that $\vec{r}_i = \vec{r}_{i,\|} + \vec{r}_{i,\perp}$. This way, we resolved the vector $\vec{r}_i$ into a part that is parallel to the rotation axis and a part that is perpendicular (see Figure XVIII.1). The distance that we desire is $|\vec{r}_{i,\perp}|$ and so

$$D_i = |\vec{r}_{i,\perp}| = \sqrt{\vec{r}_{i,\perp} \cdot \vec{r}_{i,\perp}}$$

which, on squaring both sides, gives the above result. Similarly

$$D^2 = [\vec{r} - (\vec{r} \cdot \hat{\omega})\hat{\omega}] \cdot [\vec{r} - (\vec{r} \cdot \hat{\omega})\hat{\omega}]$$

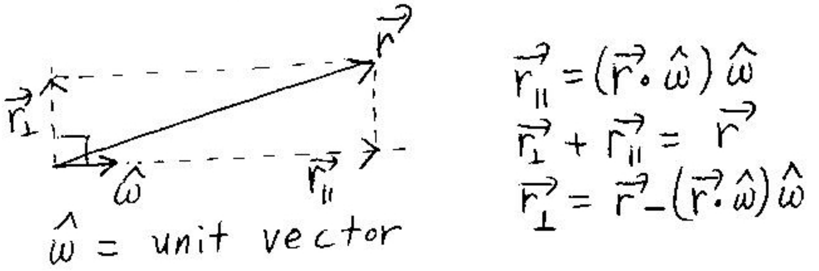


FIGURE XVIII.1. Resolution of a vector $\vec{r}$ into $\vec{r}_\|$ and $\vec{r}_\perp$ given a reference unit vector $\hat{\omega}$.

18.2. DEFINITION. **Center of Mass**

For an object that consists of discrete points (m_i and $\vec{r}_i$), the center of mass coordinate $\vec{R}_{cm}$ is defined as

$$\vec{R}_{cm} \equiv \frac{\sum_i m_i \vec{r}_i}{M}$$

where M is the total mass $M = \sum_i m_i$. Similarly, for a continuous object

$$\vec{R}_{cm} \equiv \frac{\int dm \vec{r}}{M}$$

where $M = \int dm$.

18.3. THEOREM. ***Parallel Axis Theorem***

Consider a rotational inertia I around an axis. Suppose that by translating by distance d that axis can be made to pass the center of mass. Let I_{cm} be the rotational inertia around the new translated axis passing through $\vec{R}_{cm}$. Then

$$I = I_{cm} + Md^2$$

where M is the total mass.

PROOF. [optional] It suffices to show this in the discrete object case. The proof for the continuous object case is essentially the same since the integral is the limit of the sum. Since the two axes are parallel, a single $\hat{\omega}$ (defined in Note 18.1) can be used for both axes. Let C_{cm} be the coordinate system whose origin is $\vec{R}_{cm}$. Let C be the coordinate system whose origin O lies on the first rotation axis. Let $\vec{d}$ be the vector connecting O to $\vec{R}_{cm}$, and by our choice of O (which is always possible) $\vec{d}$ is made perpendicular to $\hat{\omega}$, and thus $\vec{d} \cdot \hat{\omega} = 0$ and $d = |\vec{d}|$. From Eq. xviii.1,

$$I = \sum_i m_i D_i^2 = \sum_i m_i \left[\vec{r}_i - (\vec{r}_i \cdot \hat{\omega})\,\hat{\omega}\right] \cdot \left[\vec{r}_i - (\vec{r}_i \cdot \hat{\omega})\,\hat{\omega}\right] \tag{xviii.2}$$

where $\vec{r}_i$ is the position vector defined in C. Let us denote the position vector in the reference frame C_{cm} as $\vec{r}_{i,cm}$. Then, $\vec{r}_i = \vec{d} + \vec{r}_{i,cm}$. Inserting this into Eq. xviii.2, we get

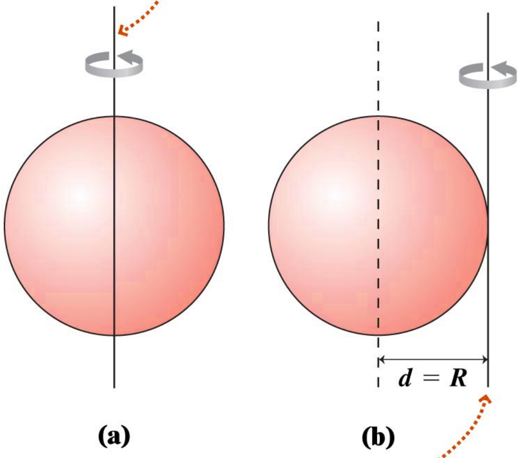
$$I = \sum_i m_i \left[\vec{d} + \vec{r}_{i,cm} - (\vec{r}_{i,cm} \cdot \hat{\omega})\,\hat{\omega}\right] \cdot \left[\vec{d} + \vec{r}_{i,cm} - (\vec{r}_{i,cm} \cdot \hat{\omega})\,\hat{\omega}\right]$$

where $\vec{d} \cdot \hat{\omega} = 0$ has been used already. Note that in the reference frame C_{cm}, $\sum_i m_i \vec{r}_{i,cm}=0$ since the center of mass is the origin in C_{cm}. This means that

$$J \equiv \sum_i m_i \vec{d} \cdot \left[\vec{r}_{i,cm} - (\vec{r}_{i,cm} \cdot \hat{\omega})\,\hat{\omega}\right] = 0$$

Since $I = \sum_i m_i \vec{d} \cdot \vec{d} + 2J + \sum_i m_i \left[\vec{r}_{i,cm} - (\vec{r}_{i,cm} \cdot \hat{\omega})\,\hat{\omega}\right] \cdot \left[\vec{r}_{i,cm} - (\vec{r}_{i,cm} \cdot \hat{\omega})\,\hat{\omega}\right]$, we get $I = Md^2 + I_{cm}$, as we set out to prove. ☐

An example that illustrates this important theorem is shown in the following picture.



This axis is through the sphere's center, so $I = \frac{2}{5}MR^2$.

This parallel axis is a distance $d = R$ away from the original axis, so $I = \frac{2}{5}MR^2 + Md^2 = \frac{7}{5}MR^2$.

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