LECTURE XVII

Rotational Motion

We have had some experience with circular motions, especially *uniform* circular motions. In this lecture, we consider circular motions that are not necessarily uniform. That is, v (and thus ω) can change as a function of time.

What we do here falls into the category of the mechanics of "rigid body," in the sense that the shape of the object that we study is assumed to be constant. The object itself is not necessarily a point particle but possibly an extended object consisting of many parts ("particles"). What can an extended object do while keeping its shape? It can move around or it can spin ("rotate") by itself. The moving around can also involve a circular motion, thus a rotation. It is all these rotations that is the subject here, while some non-circular motions (projectile motion) were the subject of previous lectures.

17.1. DEFINITION. **Angular Velocity**

$\omega = \frac{d\theta}{dt}$ (instantaneous angular velocity) and $\bar{\omega} = \frac{\Delta\theta}{\Delta t}$ (average angular velocity)

There is really nothing new here, except that we now call ω (remember, it is "omega" not "double-u") angular "velocity" rather than "speed." In what sense, is ω a vector? Let us assume that we use the convention that θ increases as we go counter-clockwise (CCW). Suppose now that there is a rotational motion going clockwise (CW). In that case $\omega = d\theta/dt$ would be negative! So the sign matters. Thus, ω must be a vector. This is in precise analogy with the one dimensional kinematics treated in early lectures: dx/dt is a vector since it can be positive or negative.

Having said this, it is rather rare to use a negative ω value. Instead, the positive direction of θ is *defined* as CW or CCW depending on whether the motion under investigation is CW or CCW, respectively, and if this convention is followed then ω is always positive. In this case, knowing the vector nature of ω is knowing which way the rotation is going. We will give a more elaborate geometrical definition of $\vec{\omega}$ later, but for now this is all we need. It is under this convention that we use ω in the following definition, not $|\omega|$.

17.2. DEFINITION. **Linear Speed and Angular Speed**

$v = \omega r$ if v is the linear speed, ω is the angular speed, and r is the radius of the circular motion

There is nothing really new here so far, compared to what we covered already in our discussions on the "uniform circular motion" (Lec_05 or Chapter 3). However, for a general circular motion, there are two types of accelerations, not just one, for $\vec{v}$.

17.3. DEFINITION. **Angular Acceleration**

$$\alpha \equiv \frac{d\omega}{dt}$$

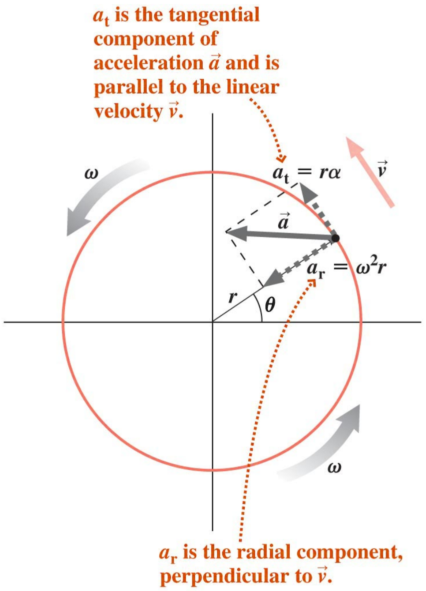
17.4. DEFINITION. **Two types of acceleration in a circular motion**

$$a_t = \frac{dv}{dt} = r\frac{d\omega}{dt} = r\alpha \qquad \text{(xvii.1)}$$

$$a_r = v\omega = r\omega^2 = \frac{v^2}{r} \qquad \text{(xvii.2)}$$

a_t is the **tangential acceleration** and a_r is the **radial (centripetal) acceleration**.

This definition is motivated by the fact that, in a general circular motion, not only $\vec{v}$ can change its direction (due to the radial acceleration), but also it can change its magnitude (due to the tangential acceleration). For the Earth going around the Sun in a circle, the latter is not relevant: the Sun is pulling the earth and that causes a finite a_r but zero a_t. However, imagine that you are on a swing, and you are pumping it faster and faster. Or, you rotate a bucketful of water with a rope attached to it, faster and faster (without spilling water!). In these two cases, both a_t and a_r are relevant.



a_t is the tangential component of acceleration $\vec{a}$ and is parallel to the linear velocity $\vec{v}$.

a_r is the radial component, perpendicular to $\vec{v}$.

Note that a **circular motion** can be considered a **one-dimensional motion**. Why? The motion is confined to a circle, which is a one-dimensional object. The "position" on a circle is uniquely specified by the angle θ. And the "velocity" for the coordinate θ is $\frac{d\theta}{dt}$, and so on. Accordingly, the results for the constant acceleration in one dimension (Lec_02, Lec_03) can be re-used if we just substitute $x \to \theta$, $v = \frac{dx}{dt} \to \omega$, and $a \to \alpha$, as in the following table.

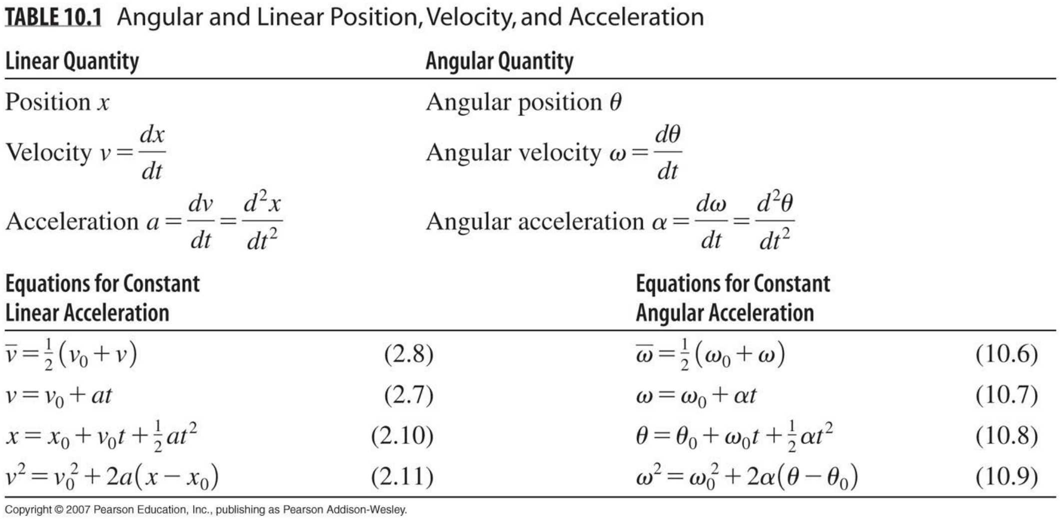


TABLE 10.1 Angular and Linear Position, Velocity, and Acceleration

Linear Quantity	Angular Quantity
Position x	Angular position θ
Velocity $v = \dfrac{dx}{dt}$	Angular velocity $\omega = \dfrac{d\theta}{dt}$
Acceleration $a = \dfrac{dv}{dt} = \dfrac{d^2x}{dt^2}$	Angular acceleration $\alpha = \dfrac{d\omega}{dt} = \dfrac{d^2\theta}{dt^2}$

Equations for Constant Linear Acceleration		Equations for Constant Angular Acceleration	
$\overline{v} = \frac{1}{2}(v_0 + v)$	(2.8)	$\overline{\omega} = \frac{1}{2}(\omega_0 + \omega)$	(10.6)
$v = v_0 + at$	(2.7)	$\omega = \omega_0 + \alpha t$	(10.7)
$x = x_0 + v_0 t + \frac{1}{2}at^2$	(2.10)	$\theta = \theta_0 + \omega_0 t + \frac{1}{2}\alpha t^2$	(10.8)
$v^2 = v_0^2 + 2a(x - x_0)$	(2.11)	$\omega^2 = \omega_0^2 + 2\alpha(\theta - \theta_0)$	(10.9)

17.5. DEFINITION. **Torque $\vec{\tau}$**

When a force $\vec{F}$ is applied at a position $\vec{r}$, the torque $\vec{\tau}$ is defined as

$$\tau = rF\sin\theta = rF_t \qquad \text{(xvii.3)}$$

(where $r = |\vec{r}|$, $F = |\vec{F}|$, and θ is the angle between $\vec{F}$ and $\vec{r}$) and the direction of $\vec{\tau}$ is determined as either CW or CCW depending on how the force is applied (more exact definition to come later). In the second expression, $F_t \equiv F\sin\theta$ is the tangential component of $\vec{F}$, in the sense that it is perpendicular to the radial vector $\vec{r}$.

Note that the position vector $\vec{r}$ enters the definition of $\vec{\tau}$. However, we observed [in Lec_04] that the origin of our coordinate system is *arbitrary*. This remains absolutely true. It just happens that the torque $\vec{\tau}$ (and other quantities, such as angular momentum, associated with the rotational motion) is *dependent* on the choice of the origin. When there is a rotational motion, obviously the center of the rotation is a very special point, and so that is the most sensible choice for the origin. Keep in the back of your mind, though, that *in principle* $\vec{\tau}$ can be defined relative to any point in space.

17.6. DEFINITION. **Rotational Inertia**

Consider an object (**a discrete object**) that consists of discrete points (each of which is described by mass m_i and position vector $\vec{r}_i$). Suppose that this object rotates around an axis, "**rotation axis**." Then, the rotational inertia of this object is defined as

$$I \equiv \sum_i m_i D_i^2$$

where D_i is the distance from the position $\vec{r}_i$ to the rotation axis.

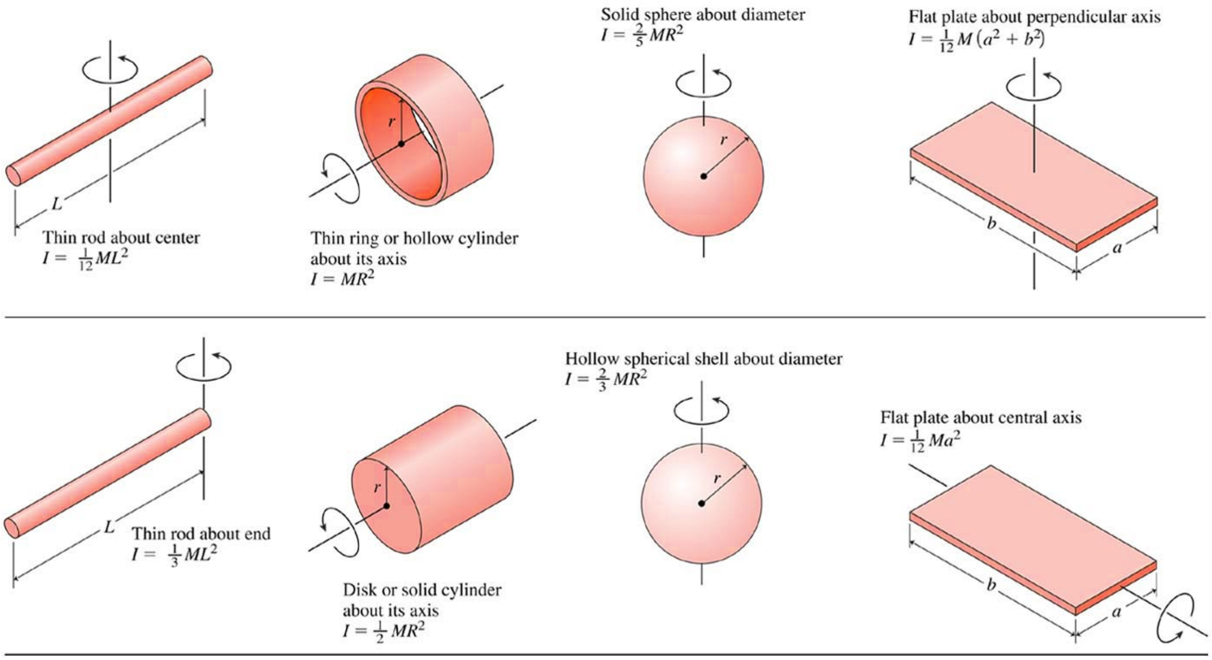
Consider an object (**a continuous object**), e.g. a sphere or a cylinder, which consists of a continuous distribution of matter. Using the technique of integral calculus, we can use the above definition to obtain the rotational inertia of this continuous object:

$$I = \int D^2 dm$$

where D is the distance between the "mass element" dm and the rotational axis.

The following table shows some well-known values of rotational inertias. All of these values can be obtained straight from integral calculus.

TABLE 10.2 Rotational Inertias



Thin rod about center
$I = \frac{1}{12}ML^2$

Thin ring or hollow cylinder about its axis
$I = MR^2$

Solid sphere about diameter
$I = \frac{2}{5}MR^2$

Flat plate about perpendicular axis
$I = \frac{1}{12}M(a^2 + b^2)$

Thin rod about end
$I = \frac{1}{3}ML^2$

Disk or solid cylinder about its axis
$I = \frac{1}{2}MR^2$

Hollow spherical shell about diameter
$I = \frac{2}{3}MR^2$

Flat plate about central axis
$I = \frac{1}{12}Ma^2$