LECTURE XVI

Conservation of Energy

We need to distinguish between a "conservative force" and a "non-conservative force."

16.1. Definition. **A conservative force gives back** the work done against it **without subtraction or addition**.

The above definition is very physical and intuitive, but it is also very loose. We need a mathematical definition. Here is a good one.

16.2. Definition. **Conservative force**
$\vec{F}$ is a conservative force if $\oint \vec{F} \cdot d\vec{r} = 0$ for *any* closed path. Note that $\oint$ is the symbol for integral over a closed path.

Here is a brief discussion of how this rigorous definition is motivated. Consider the following process, which is overall a very very slow ("adiabatic") process. Imagine an object (e.g. a baseball) under the influence of a gravitational force $\vec{F}_g$. You do the following to the object.

1. You keep the velocity of the object constant at $\delta\vec{v}$ (pointing up) and raise the object up by a certain height h. Due to the assumption of a very small $\delta v \equiv |\delta\vec{v}|$, given in the next step, this process is very very slow. You are infinitely patient! [This is the step of you doing work against the gravity.]
2. You reverse the direction of the object to $-\delta\vec{v}$. We assume that $\delta v \equiv |\delta\vec{v}|$ is so small that this reversal of direction can be done in such a manner that the change of position involved in this step is negligible compared to h in step 1. (This is always possible.)
3. You keep the velocity of the object constant, this time at $-\delta\vec{v}$, and bring it back to the original position.

From the previous lecture ("work–energy theorem"), we know that the net work done on the object is equal to the change of its kinetic energy. Since the object starts and ends at the same speed (δv), there is no net work done to it at all, from step 1 through step 3. This is easy to understand, since in the above steps, any work is done only in steps 1 and 3, and the net force is always zero there. But, there is more! For step 1, we have $\int_{step\,1}(\vec{F}_g + \vec{F}_H) \cdot d\vec{r} = 0$, since $\vec{F}_H = -\vec{F}_g$ due to Newton's 2nd law. Thus, we have $\int_{step\,1}(\vec{F}_g - \vec{F}_g) \cdot d\vec{r} = 0$. But wait a minute! The second integral $\int_{step\,1} -\vec{F}_g \cdot d\vec{r}$ can be thought of as $\int_{step\,1} \vec{F}_g \cdot (-d\vec{r})$, namely keeping the same force but reversing the path! Namely, the second integral is equal to $\int_{step\,2} \vec{F}_g \cdot d\vec{r}$. Thus, we get $\left(\int_{step\,1} + \int_{step\,2}\right) \vec{F}_g \cdot d\vec{r} = 0$. That is, $\oint \vec{F}_g \cdot d\vec{r} = 0$, for the closed path from step 1 through step 3. While this argument is made only for a linear path here, it can be generalized to the case of any path with the help of multi-variable calculus that I won't go into, and thus the definition.

What about a non-conservative force? Where does the argument above break down? Consider a frictional force, for example. The difference is the following. In the above argument, it was important that the gravitational force remains the same for the same point in space even when the path is reversed. This is no longer true for a frictional force. Why? Because a frictional force is dependent on the velocity (in the case of the kinetic friction) or on the applied force (in the case of the static friction)! With this in mind, let us consider similar steps like the above for a frictional force. For instance, on a horizontal surface with friction, we (1) push an object from position 1 to position 2, (2) turn it around, and then (3) push it back from position 2 to position 1. Again step 1 and step 3, very very slowly. The difference here is that when we reverse the path, the frictional force also changes its direction! Therefore, $\oint \vec{F} \cdot d\vec{r} < 0$, *not* zero, for a frictional force. Other non-conservative forces have similar properties. Namely, they are not uniquely determined by the position vector, but dependent on other parameters such as velocity or time. [Not all such forces are necessarily non-conservative, though.]

So, what is going on here? In the case of a conservative force, the work done by hand in step 3 of the motion is negative and exactly the same in magnitude as the work done by hand in step 1, since the conservative force "gives back" in equal amount. In the case of a frictional force, the work done by hand in step 3 remains positive, since a frictional force always "takes without giving back." For a frictional force, the work that you put in is "lost," because it is transformed into something else ("heat") than the mechanical energy of the the object. Generally, a non-conservative force does not have to be a mean taker. It can also be an over-giver. For instance, a time-dependent external electromagnetic field (which we will not consider in this course) can be such an over-giver.

16.3. DEFINITION. **Potential Energy**

$\Delta U_{AB} \equiv U_B - U_A = -\int_A^B \vec{F} \cdot d\vec{r}$, where $\vec{F}$ is a conservative force, and A and B are two points in space, is the change of the potential energy from point A to point B.

16.4. NOTE. The potential energy has the following important properties to keep in mind.

- ΔU_{AB} is work done against the conservative force $\vec{F}$.
- ΔU_{AB} is dependent only on end points A and B, not the actual path of integration in $-\int_A^B \vec{F} \cdot d\vec{r}$ (due to Definition 16.2).
- The potential energy U is defined only in relative terms (through the difference between U_A and U_B). The absolute value of the potential energy is meaningless and can be fixed to any value for a given problem. However, there are often-used conventions, as in the following examples.

16.5. EXAMPLE. **Potential energy for a constant gravitational acceleration**

Near the earth surface, $\vec{F}_g = -mg\hat{j}$, and thus $\Delta U_{AB} = -\int_A^B \vec{F}_g \cdot d\vec{r} = mg \int_A^B y = mg\Delta y$. Thus, if we use the often-used convention $U(y = 0) = 0$, then

$$U(y) = mgy$$

16.6. EXAMPLE. Potential energy of a spring [**Elastic potential energy**]

Consider a spring with one end fixed. We use the coordinate x for the position of the other end. I.e., the spring lies along the x axis (which *can* be the vertical axis, if you like). Define $x = 0$ for spring at equilibrium. Then, $\vec{F}_x = -kx\hat{i}$. $\Delta U_{AB} = \int_A^B kxdx = \frac{1}{2}k\Delta(x^2)$.

Thus, if we use the often-used convention $U(x=0)=0$, then

$$U(x) = \frac{1}{2}kx^2$$

16.7. THEOREM. ***Conservation of Mechanical Energy***

If all forces acting on an object are conservative, then its ***mechanical energy, defined as***

$$K + U$$

where K is its kinetic energy ($mv^2/2$) and U is its total potential energy (sum of all potential energies for all conservative forces acting on it), is conserved, i.e. the value of the mechanical energy does not change during the course of the motion of the object. In this sense, $K+U$ is called a ***constant of motion****.*

PROOF. Consider one conservative force acting on the object, as the consideration of more than one conservative force does not modify the proof in any essential way. Due to the work energy theorem, $K_B - K_A = \int_A^B \vec{F} \cdot d\vec{r}$, where A and B correspond to the end points (position A and position B) of a motion under consideration. By definition 16.3, the right hand side is $U_A - U_B$. Thus, we get $K_B - K_A = U_A - U_B$, which means

$$K_A + U_A = K_B + U_B$$

which completes the proof. □

16.8. EXAMPLE. For a **projectile motion**, the mechanical energy conservation means

$$\frac{1}{2}mv_1^2 + mgy_1 = \frac{1}{2}mv_2^2 + mgy_2 \tag{xvi.1}$$

is a constant of motion, where $v^2 = v_x^2 + v_y^2$. For a **one dimensional motion with a constant acceleration** a, the work-energy theorem means
$ma(x_2 - x_1) = \frac{1}{2}m(v_2^2 - v_1^2)$ which can be summarized as

$$v_2^2 - v_1^2 = 2a(x_2 - x_1) \tag{xvi.2}$$

which is the equation that was not recommended for memorization in Chapter 2 [now you know why]. Notice that these two equations are closely related to each other, although the latter equation is derived from the work-energy theorem and is thus valid for both conservative and non-conservative forces. The question that came up again and again in previous chapters is the following: when a ball is thrown up at initial speed v_0, how high does it go? From the mechanical energy conservation, the solution is obtained very trivially: $\frac{1}{2}mv_0^2 = mgh$, and thus $h = \frac{v_0^2}{2g}$. This answer happens to be applicable to the stopping distance problem of a car, if we replace g with the deceleration due to friction, d. This is equivalent to using Eq. xvi.2 with $a = -d$.

16.9. THEOREM. ***Force from potential energy***

For a given potential energy $U(\vec{r})$, the corresponding conservative force $\vec{F}$ is given by

$$F_x = -\frac{dU}{dx}$$

in one dimension. In higher dimensions, this result is generalized as $F_x = -\frac{\partial U}{\partial x}$, $F_y = -\frac{\partial U}{\partial y}$, etc., i.e. $\vec{F} = -\vec{\nabla}U$.

PROOF. Immediate consequence of $U_B - U_A = -\int_A^B \vec{F} \cdot d\vec{r}$. For the one dimensional case, please see Appendix D, if necessary. For higher dimensional cases, the proof is essentially the same. Choose the path from A to B as a path along the x direction to prove $F_x = -\frac{\partial U}{\partial x}$, and a path along the y direction to prove $F_y = -\frac{\partial U}{\partial y}$ and so on and so forth. □