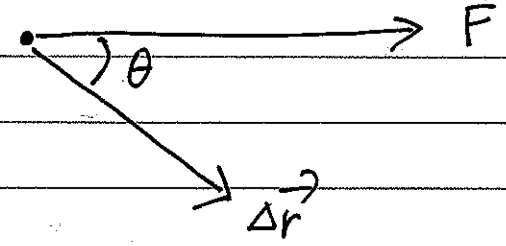
§1. Work $\begin{pmatrix} \text{small displacement } \Delta\vec{r} \\ \text{or constant force } \vec{F} \end{pmatrix}$

You apply force $\vec{F}$ to an object.
Suppose the object moved by $\Delta\vec{r}$.



(during a small time interval Δt)

The work that you did on the object

$$W \equiv |\vec{F}| \cdot |\Delta\vec{r}| \cos\theta$$

(Recall that $|\vec{A}|$ = length of vector $\vec{A}$)

Note ① This is the so-called (dot product / innerproduct) of two vectors

$$\vec{A} \bullet \vec{B} \equiv |\vec{A}|\,|\vec{B}| \cos\theta$$

θ : angle between the two vectors $\vec{A}, \vec{B}$

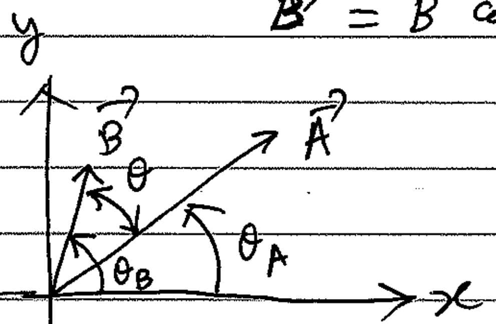
If $\vec{A} = A_x \hat{i} + A_y \hat{j}$, $\vec{B} = B_x \hat{i} + B_y \hat{j}$

$$\vec{A} \bullet \vec{B} = A_x B_x + A_y B_y$$

Easily generalized to any dimension not just 2-dim

proof) $\vec{A} = A \cos\theta_A \hat{i} + A \sin\theta_A \hat{j}$, $A = |\vec{A}|$

$\vec{B} = B \cos\theta_B \hat{i} + B \sin\theta_B \hat{j}$, $B = |\vec{B}|$



$$A_x B_x = AB \cos\theta_A \cos\theta_B$$

$$A_y B_y = AB \sin\theta_A \sin\theta_B$$

$$A_x B_x + A_y B_y = AB(\cos\theta_A \cos\theta_B + \sin\theta_A \sin\theta_B)$$

$$= AB \cos(\theta_A - \theta_B)$$

$$= AB \cos\theta \qquad \text{QED}$$

Appendix B (B.S. FACT)

$\theta_A - \theta_B = \pm\theta$

Note ② Work can be positive, zero or negative.

Positive : You push a car and it goes as you wish.
Zero : You are carrying a heavy bag and walk on the street at a constant velocity.
(Non-intuitive but true! You are not doing any work to the bag, only to your muscles!)
Negative : You are trying to keep your backpack by pulling it, but a giant bear is pulling that, and you, in the opposite direction.
(You are doing a negative work on the backpack! You are making it harder for the bear to get the bag...)

In short, we come back to a similar maxim that we encountered in regards to $\vec{v}$ and $\vec{a}$ ($\vec{v}$ and $\vec{a}$ are apple and orange!):

$\vec{F}$ (force) and $\Delta\vec{r}$ (displacement) are not necessarily in the same direction!!

The work done by force $\vec{F}$ (not necessarily the net force) on an object is

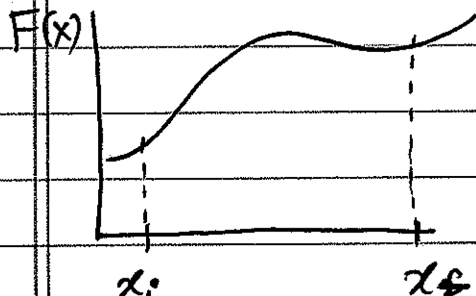
$$\vec{F} \cdot \Delta\vec{r}$$

which basically measures the "effectiveness" of force $\vec{F}$ in causing the displacement in its ($\vec{F}$'s) direction.

§2. Work (large displacement $\vec{r_1} \to \vec{r_2}$ with possibly non-constant force $\vec{F}$)

There is nothing to fear here. We just have to consider the ideas of the calculus of integrals and feel comfortable with them.

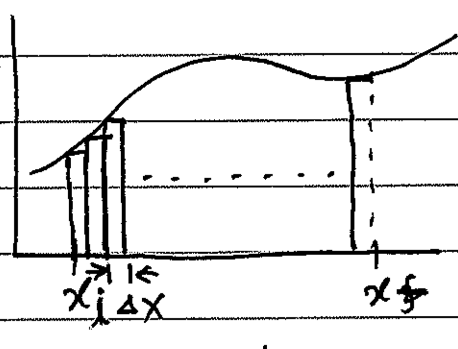
§2.1 Case 1) 1 dimension



What's the work done from x_i to x_f?

Key idea of calculus: (divide and conquer)

If we divide up the interval $[x_i, x_f]$ into many many subintervals, then in each subinterval we can apply the ~~idea~~ results of the previous section (§1) since $F(x) \approx$ constant.



$$x_n = x_i + n \cdot (\Delta x)$$

$$\Delta x = \frac{x_f - x_i}{N}$$

$n = 0, \ldots, N-1$

$$W_N \approx \sum_{n=0}^{N-1} F(x_n)\,\Delta x$$

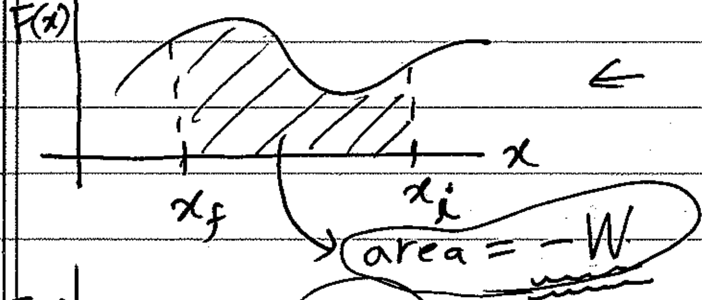
→ The sum of ares of rectangles.

As $N \to \infty$, ~~the sum~~ $W_N \to$ the area under the curve $F(x)$ between x_i and x_f

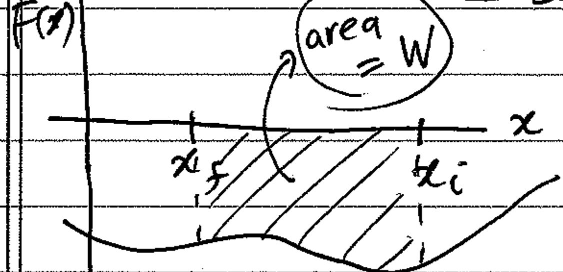
integral of $F(x)$ from x_i to x_f - - - → $\equiv \boxed{\int_{x_i}^{x_f} dx\, F(x) = W}$

$$W = \int_{x_i}^{x_f} F(x)\,dx$$ main result

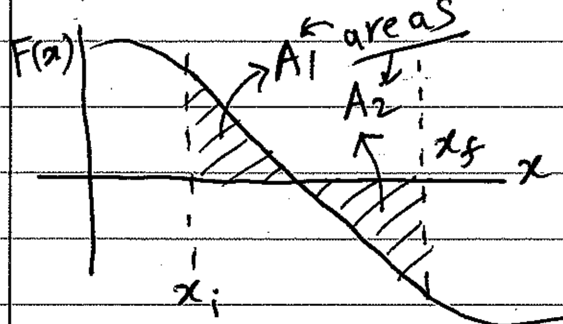
Note ① $x_i < x_f$, $F(x) > 0$ was considered but this doesn't have to be true.



$\leftarrow W < 0$

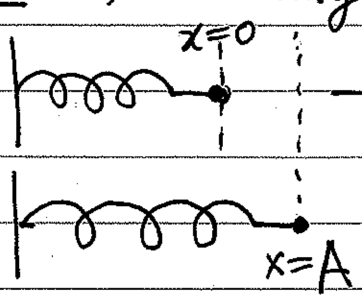


$\leftarrow W > 0$



$\leftarrow W = 0$ if $A_1 = A_2$

Note ② Stretching a spring (fixed at one end)



$F_s = -kx$ Hooke's law

Now x can be taken as the position coordinate of the moving end point because (generally x = length change rel. to normal length)

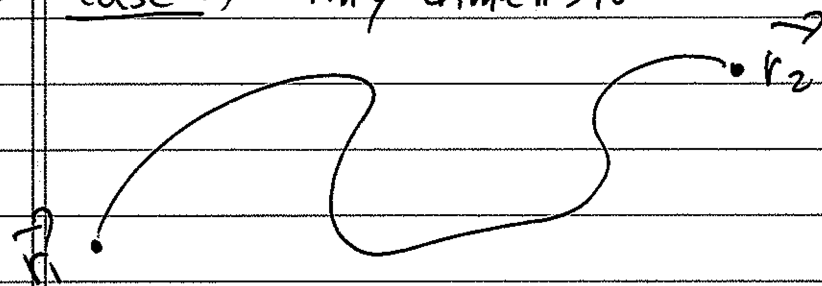
Say, you stretch the string very very slowly from $x=0$ to $x=A$, the work that you did?

$F_H = kx$ (positive direction) ← Newton 3rd law

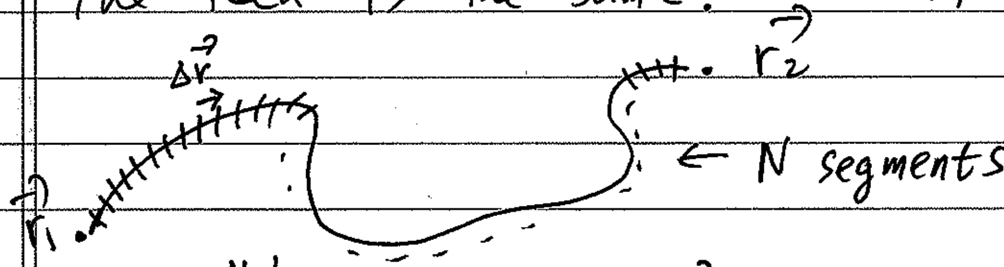
$$\boxed{W_H = \int_0^A F_H(x)\,dx = \int_0^A kx\,dx = \frac{1}{2}kA^2}$$!

Taking A as variable x → $\boxed{W_H(x) = \frac{1}{2}kx^2}$

§2.2 <u>Case 2)</u> Any dimension



The idea is the same. Divide and conquer.



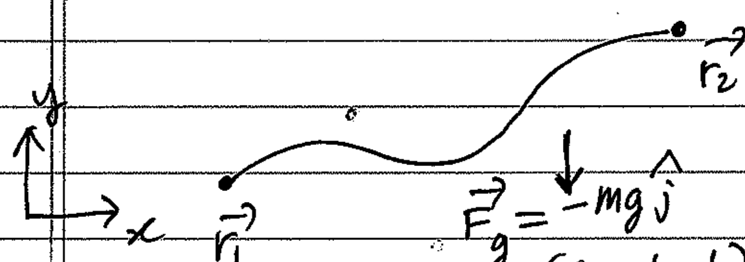
$$W_N \approx \sum_{n=0}^{N-1} \vec{F}_n \cdot \Delta\vec{r}_n$$

$\vec{F}_n$: approximate constant force for each segment $\Delta\vec{r}_n$

As $N \to \infty$, we get a line integral

$$W = \lim_{N\to\infty} W_N = \int_{\vec{r}_1}^{\vec{r}_2} \vec{F}(\vec{r}) \cdot d\vec{r}$$

<u>Example</u>) Work done by gravity



$\vec{F}_g = -mg\hat{j}$ (constant)

$$W_g = \int_{\vec{r}_1}^{\vec{r}_2} \vec{F}_g \cdot d\vec{r} = -mg\int_{\vec{r}_1}^{\vec{r}_2} \hat{j} \cdot d\vec{r}$$

$$d\vec{r} = dx\,\hat{i} + dy\,\hat{j}$$

$$\hat{j} \cdot d\vec{r} = dy$$

$$\boxed{W_g = -mg\int_{\vec{r}_1}^{\vec{r}_2} dy = -mg\,(y_2 - y_1)}$$

☆☆