LECTURE V

Motion in 2D, 3D (cont.)

V.1. Projectile Motion

Consider a ball, or a rocket, moving close to the surface of the earth, under the influence of gravity. The best way to take the coordinate system is to take the direction of the horizontal motion of the ball as the x direction and the vertical direction as the y direction. If we do that, then this motion becomes a 2D problem, even though it happens in 3D. Furthermore, close to the surface of the earth, the gravity can be described in terms of an acceleration vector $\vec{a} = (0, -g)$, where $g \approx 9.81$ m/s^2. Applying the solutions obtained in the previous section, by plugging in $a_x = 0$ (nice!) and $a_y = -g$, we obtain

$$\begin{aligned} v_x &= v_{x,0} \\ x &= x_0 + v_{x,0} t \end{aligned} \tag{v.1}$$

and

$$\begin{aligned} v_y &= v_{y,0} - gt \\ y &= y_0 + v_{y,0} t - \frac{1}{2} g t^2 \end{aligned} \tag{v.2}$$

It is important to remember these equations in plain terms, as follows.

5.1. FACT. ***Essential physics of projectile motion***

For the projectile motion near the surface of the earth $\vec{a} = -g\hat{j} = (0, -g)$. The motion along the x axis is that of a constant velocity and the motion along the y axis is the same as a ball tossed up vertically.

This is neat! On the other hand, the fact that the two independent motions along the x and y axes happen at the same time means that one can ask the question like "Derek Jeter throws the ball to the first base. When the ball is at the maximum height, how much distance did it travel?" Let me state the following simple fact in this connection.

5.2. FACT. ***Same height, same time***

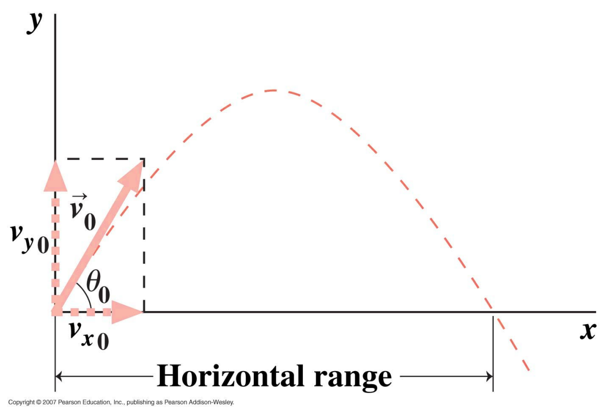
Say one throws a ball, and the ball goes forward while goes up and comes down at the same time. The total time that it comes down to the same height is $2\sqrt{2h/g}$, where h is the maximum height that the ball travels, relative to the initial height. That is, regardless of other details, the two trajectories with the same height take the same amount of time.

PROOF. The natural coordinate system to use is such that $y = 0$ initially. The ball goes up to $y = h$ and then comes down to $y = 0$ again. Say t_m corresponds to the time at which the ball reaches the maximum height. By definition, the maximum height must correspond to $v_y = 0$, since the ball is not going up any further. So, we have two equations for $t = t_m$: $h = 0 + v_{y0} t_m - \frac{1}{2} g t_m^2$ and $0 = v_{y0} - g t_m$. The latter equation means $v_{y0} = g t_m$, and plugging this into the first equation, we get $h = \frac{1}{2} g t_m^2$. Thus, $t_m = \sqrt{\frac{2h}{g}}$.

It is no coincidence that this is precisely the time it takes a ball to fall distance h when released freely, corresponding to the second half of the y motion. Thus, the answer that we are looking for is $2t_m = 2\sqrt{2h/g}$. □

OK, let us summarize. What we did is to break down the y motion into two parts. The second part is a free fall motion, while the first part is the same exact free fall, viewed backwards (time-inverted free-fall). To remember this broken-down view is vastly more important than to remember the end result of Fact 5.2. The same reasoning leads to the following well known fact.

5.3. FACT. ***Optimum angle of a throw***
To throw a ball to the maximum distance, throw it at 45 degrees or $\pi/4$.



PROOF. Again, we define the coordinate system so that $x_0 = y_0 = 0$. Since there is no acceleration along the x axis, the distance the ball travels is $d = v_{x,0}t_{total}$ (which the textbook calls the "range of a projectile"). For the y axis motion, v_y goes from 0 to $-v_{y,0}$ with acceleration $-g$ in time $t_{total}/2$ (here we are considering the 2nd half of motion) and so $-v_{y,0} = -gt_{total}/2$, which means $t_{total} = 2v_{y,0}/g$. Plugging this into the equation for d, we get $d = 2v_{x,0}v_{y,0}/g$. [OK, at this point, I would check the units, or the dimensions, to make sure that I did not make any silly mistake. Dimension-wise, $[\text{RHS}] = [v^2/g] = [(L/T)^2/(L/T^2)] = [L]$, which is correct! RHS = right hand side.] In terms of angle θ_0 of throw and speed v_0, we have $v_{x,0} = v_0 \cos\theta_0$ and $v_{y,0} = v_0 \sin\theta_0$. Thus, $d = 2v_0^2 \cos\theta_0 \sin\theta_0/g$. Using a well-known trigonometric identity, $2\sin\theta_0\cos\theta_0 = \sin(2\theta_0)$ (the textbook has a typo here), we have

$$d = v_0^2 \sin(2\theta_0)/g$$

To maximize d for a given v_0, the optimum angle is $\theta_0 = 45°$, at which $\sin(2\theta_0)$ is the maximum possible value = 1. □

This is the case when there is no air resistance, of course. What happens if you do include air resistance? In this case, numerical solutions need to be consulted. The answer is that the angle needs to be smaller than 45 degrees. Let us end with a simple fact for a constant acceleration motion.

5.4. FACT. ***The trajectory of motion in 2D for a non-zero constant acceleration is a parabola.***

PROOF. In the solutions for the projectile motion, Equations v.1,v.2, the trajectory is obtained by eliminating t. First, from, Eq. v.1, we have $t = (x - x_0)/v_{x,0}$. Plugging this into Eq. v.2, it is clear that we will get an equation that will end up in the form of $y = A + Bx + Cx^2$, where A, B, C are constants that are functions of $x_0, y_0, v_{x,0}, v_{y,0}, g$. This is proof enough, but for our satisfaction, let us explicitly evaluate A, B, C. By conveniently choosing the coordinate system so that $x_0 = 0$ and $y_0 = 0$, we have

$$y = \frac{v_{y,0}}{v_{x,0}} x - \frac{g}{2v_{x,0}^2} x^2 \tag{v.3}$$

Thus, $A = 0$, $B = \frac{v_{y,0}}{v_{x,0}}$, $C = -\frac{g}{2v_{x,0}^2}$ when $x_0 = y_0 = 0$. For finite x_0 or y_0, all we need to do is substitute $x - x_0$ for x and $y - y_0$ for y, and then $A = -\frac{v_{y,0}}{v_{x,0}} x_0 - \frac{g}{2v_{x,0}^2} x_0^2$, $B = \frac{v_{y,0}}{v_{x,0}} + \frac{g}{v_{x,0}^2} x_0$, $C = -\frac{g}{2v_{x,0}^2}$. This is the end of proof. [What is the trajectory if the constant acceleration is zero? It is a line.]

Now, you might ask, wait a sec, you just did the proof for a very specific case when $\vec{a} = -g\hat{j}$, and what if $\vec{a}$ is pointing to an arbitrary direction. In that case, it is NOT recommended to solve Equations iv.9,iv.10. That would be horrendous. The fact is that if there is a constant vector in a problem and if the question is asking about certain properties independent of the choice of the coordinate system, then it is the best strategy to define our coordinate system to point along that constant vector. Note that most, if not all, physics questions that are independent of the coordinate system that we choose to solve the problem. The reason is simple. The coordinate system is something that we draw in the air just to make computation easy, not because they are really real. This is the case here too. The shape of the trajectory is a concept that is independent of how we take the coordinate system. Namely, as long as in at least *one* Cartesian coordinate system, if the relation $y = A + Bx + Cx^2$ with certain constants A, B, C describes the trajectory, then that trajectory is a parabola. Now, of course, if there are more than one constant vectors in a problem, we would have to make the call which constant vector to choose to define an axis. In the current case, it is a no brainer. Just choose our coordinate system so that $\vec{a}$ points along the $-y$ direction. In this sense, any 2D motion with a constant $\vec{a}$ is a projectile motion, and the shape of the trajectory is a parabola. □

5.5. OBSERVATION. ***Remember how and why, not the complicated end result.***

No one is expected to, and should not, memorize the actual equation of the parabola, as presented above, or as presented in Eq. 3.14 of the text. When a problem requires it, you should use the most convenient ("the simplest") coordinate system to derive it quickly, as done above.

V.2. Circular Motion

Circular motions or approximately/instantaneous circular motions are ubiquitous – spinning tops, car on a curved ramp, planets and moons and galaxy and all. The basic kind of circular motion is the so-called "uniform circular motion."

5.6. DEFINITION. **Uniform circular motion**

A motion on a fixed circle with a constant speed.

It follows that this motion has to be periodic. Let us state some elementary facts about this important motion. More generally, the speed can change while the path is permanently or temporarily a circle. Then we call it just a "circular motion."

5.7. FACT. ***A constant acceleration cannot give rise to a circular motion.***

PROOF. This follows directly from Fact 5.4, since a circular motion is a 2D motion, while a circle is neither a parabola (non-zero constant $\vec{a}$) nor a line (zero $\vec{a}$). ☐

5.8. FACT. ***The acceleration is time-dependent for a circular motion.***

PROOF. This is an immediate consequence of the above Fact that we just proved. Later on, we will see what kind of force does give rise to a circular motion, but at the moment, this is all we have to say. ☐

5.9. FACT. ***Consider a uniform circular motion with speed v, angular speed ω and radius r. Then $v = r\omega$.***

PROOF. This is quite intuitive, if we recall the fact that the angle is measured in unit of radian in the SI unit [see Section I.2]. So, if the uniform circular motion occurs with the period T, then the angular speed is [total angle divided by time]

$$\omega = \frac{2\pi}{T} \tag{v.4}$$

and the speed is [total distance divided by time]

$$v = \frac{2\pi r}{T} \tag{v.5}$$

So, it is easily seen that

$$v = r\omega \tag{v.6}$$

In this derivation, we used the fact that in a uniform circular motion the speed is constant and so is the angular speed. However, it is instructive to derive the same result from the differential calculus point of view. For this, let us say that in time Δt the angle changes by $\Delta\theta$ as in Figure V.1. What is the angular speed? It is

$$\omega = \lim_{\Delta t \to 0} \frac{\Delta\theta}{\Delta t} \tag{v.7}$$

In Figure V.1(b), note that $\Delta l = r\Delta\theta$. We consider the limit $\Delta t \to 0$ and thus $\Delta\theta \to 0$. In this case, it is easy to see from Figure V.1(b) that $|\Delta\vec{r}| \approx \Delta l = r\Delta\theta$. Thus the speed is

$$v = \lim_{\Delta t \to 0} \left| \frac{\Delta\vec{r}}{\Delta t} \right| = \lim_{\Delta t \to 0} \frac{r\Delta\theta}{\Delta t} = r \lim_{\Delta t \to 0} \frac{\Delta\theta}{\Delta t}$$

where in the last step the fact that r is independent of t is used. Thus, $v = r\omega$. ☐

5.10. FACT. ***The acceleration for a uniform angular motion always points to the center of the circle and its magnitude is ωv.***

PROOF. This and the previous fact are certainly worth knowing by heart. Here goes the proof. In Figure V.1(d), it is indicated (as it is easy to show) that, when $\vec{r}_1$ and $\vec{r}_2$ make an angle $\Delta\theta$, $\vec{v}_1$ and $\vec{v}_2$ do the same. Thus, $|\Delta\vec{v}| \approx v\Delta\theta$, and therefore, the magnitude of the acceleration

$$a = \left| \lim_{\Delta t \to 0} \frac{\Delta\vec{v}}{\Delta t} \right| = v \lim_{\Delta t \to 0} \frac{\Delta\theta}{\Delta t} = v\omega$$

noting that v is time-independent. In terms of the direction of $\vec{a}$, note that from Figure V.1(e), $\Delta\vec{v}$ points to the center of the circle on the average for any time t, and thus the

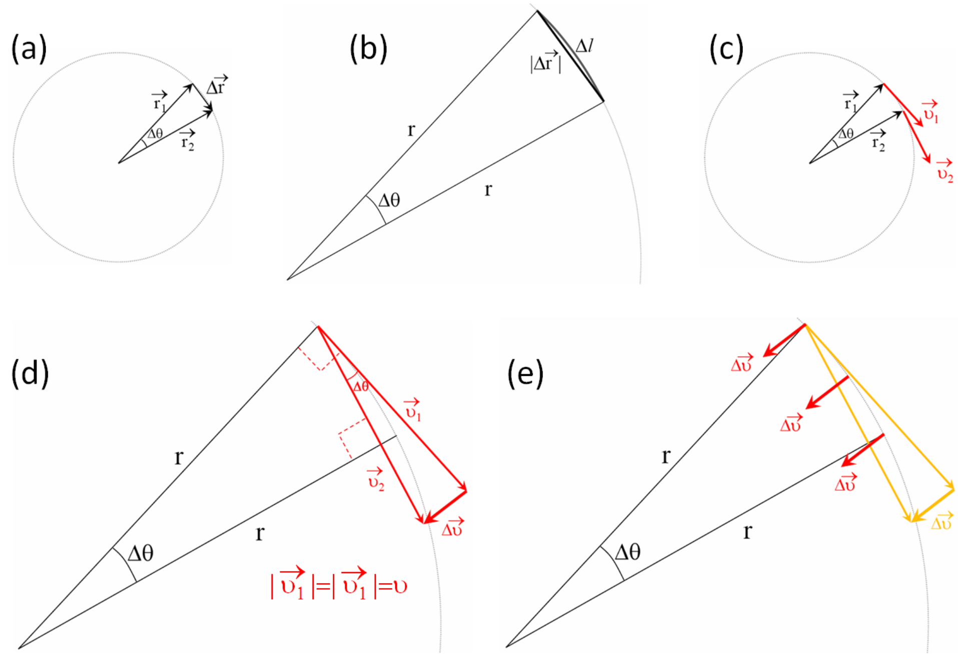


FIGURE V.1. Analysis of uniform circular motion

average acceleration $\Delta\vec{v}/\Delta t$. So, it follows that $\vec{a} = \lim_{\Delta t \to 0} \Delta\vec{v}/\Delta t$ points to the center ("centripetal" acceleration). Finally, it is important to be able to recall these expressions

$$a = v\omega = \frac{v^2}{r} = r\omega^2 \qquad \text{(v.8)}$$

□

5.11. OBSERVATION. ***Motion on a curved path at any given instant is partly a circular motion.***

Note that the consideration of a circular motion can be useful even for a non-circular motion. Be warned, though, that in general there is a tangential acceleration, in addition to the "centripetal" acceleration or radial acceleration. The radial acceleration, and only it, can be described in terms of the instantaneous speed v, the instantaneous angular speed ω, and the instantaneous radius ("radius of curvature") in exactly the same way as in the above equations. See the lower part of page 44 of the text.