LECTURE IV

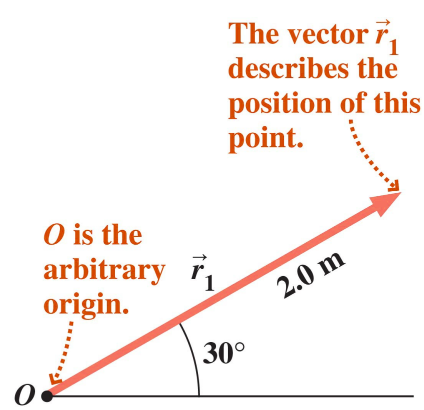
Motion in Two and Three Dimensions

IV.1. Vectors

A vector is a mathematical object that is characterized by direction as well as length. A scalar is a mathematical object that is characterized by length alone and has no sense of direction. Examples of scalar quantities are the length of a vector and the angle between two vectors. The notion of vector is introduced here in terms of position and displacement, but it is applicable to other quantities such as velocity, acceleration, force, [linear] momentum, and angular momentum. In a mundane version of more sophisticated mathematical language, vectors are objects that can be scaled (stretched, shortened, reversed) and joined (added or subtracted). Namely, they are like arrows.

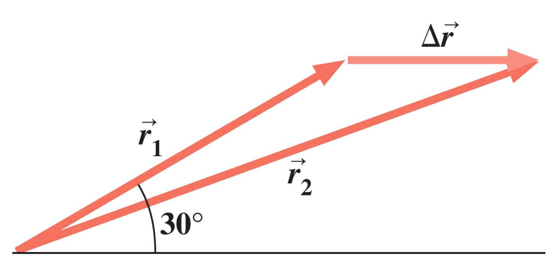
4.1. DEFINITION. **Position vector**

The basic quantity in mechanics is the displacement, the change of position. In most cases, the absolute position is not important, since one can take any arbitrary point as origin. Convenience, not law, governs our choice of the origin [1]. For a given choice of the origin, any position can be visualized as a vector, as below. Namely, *position is a vector*.



4.2. DEFINITION. **Displacement vector**

The displacement vector is the change of the position vector. In the diagram below, the displacement from $\vec{r}_1$ to $\vec{r}_2$ is defined as $\Delta\vec{r}$.

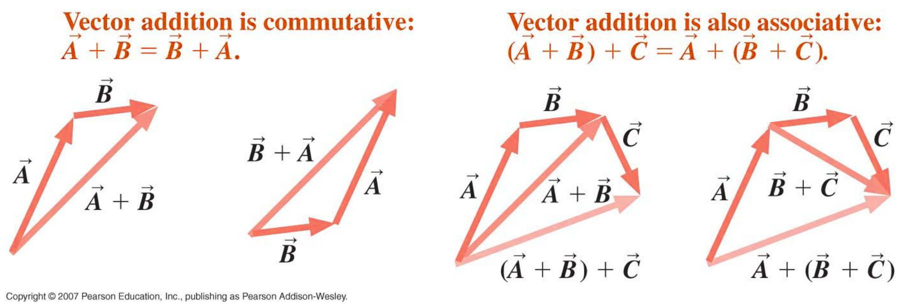


[1] The arbitrariness of the origin is a recurring theme for many physical quantities (position, time, angle, velocity, momentum, energy, etc) with the notable exceptions of, e.g. temperature and entropy.

4.3. DEFINITION. **Vector sum and difference**

The procedure of adding two (or more) vectors is to "join them head to tail." Therefore, to subtract two vectors, draw them with a common origin, and then connect the end of the second to the end of the first. According to this definition, the above diagram, defining $\Delta\vec{r}$ can be interpreted as either $\Delta\vec{r} = \vec{r}_2 - \vec{r}_1$ or $\vec{r}_2 = \vec{r}_1 + \Delta\vec{r}$.

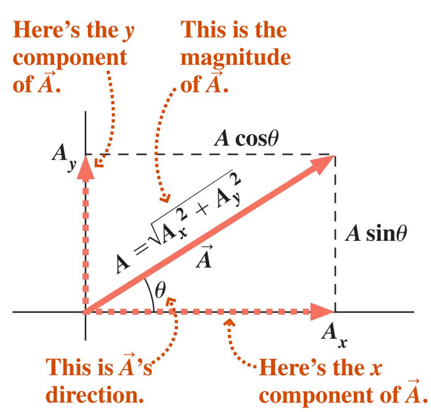
Vector addition has the following properties, much like number addition. Terms "commutative" and "associative" are rigorous mathematical terms, which you do not need to remember in this course. Keep in mind that subtraction is *not* commutative, but it remains associative, just like in number subtraction. From the above figure, one can convince oneself easily that $\vec{r}_1 - \vec{r}_2 = -(\vec{r}_2 - \vec{r}_1)$ and *not* $\vec{r}_1 - \vec{r}_2$ itself.



Another important operation that one can do on a vector is to multiply it by a number (i.e., scalar). This corresponds to stretching, shortening, and reversing, etc. Vector additions and vector-scalar multiplications have commutative and associative properties just like number additions and multiplications.

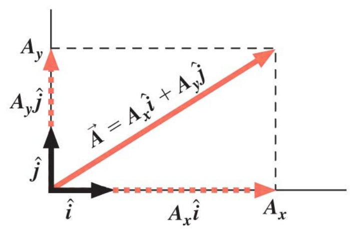
4.4. DEFINITION. **Representation of vector**

Vectors are conveniently visualized as arrows. While the arrow visualization is sufficient for understanding vectors, if we want to *do* anything useful with vectors, we need to *represent* them as a list of numbers, and this requires a coordinate system. The Cartesian coordinate system, consisting of orthogonal $x, y, \ldots$ axes, is by far the most basic and the most standard coordinate system, and students will not/rarely need to know any other coordinate systems during this course. The diagram below shows how the vector $\vec{A}$ can be *resolved* into two vectors (dashed arrows) that lie along each axis, in the sense that the two vectors add up to the original vector. The resolution is accomplished by drawing a vertical line to each axis from the tip of the original vector. The coordinate of each resolved vector along its axis is called the "component" of the original vector. The collection of components (A_x, A_y) for the two dimensional vector $\vec{A}$ is its *representation*.

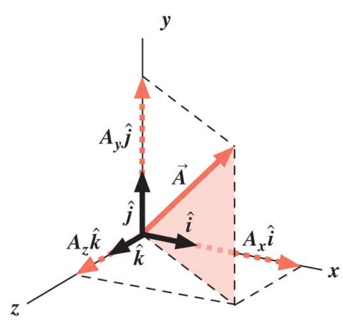


4.5. DEFINITION. **Unit vectors and a notation matter**

The vector of unit length along the positive direction of each axis is called a unit vector. Conventional notations for unit vectors in the Cartesian coordinate system are: $\hat{i}, \hat{j}, \hat{k}$, or $\hat{x}, \hat{y}, \hat{z}$ or $\hat{e}_1, \hat{e}_2, \hat{e}_3$. So, for a 2D vector, $\vec{A} = A_x\hat{i} + A_y\hat{j}$, where (A_x, A_y) are the components. Sometimes a quick notation $\vec{A} = (A_x, A_y)$ may be used to mean $\vec{A} = A_x\hat{i} + A_y\hat{j}$. Note that in this case "=" does not mean "is equal to" but means "is represented by." So, one has to be careful not to use two different coordinate system representations if one uses the notation $\vec{A} = (A_x, A_y)$.



(a)



(b)

4.6. NOTE. **Representation makes it easy to calculate.**

The reason why the coordinate system is introduced is because it makes it easy to calculate things. Vector additions, subtractions, and vector-scalar multiplications are all easily calculated component-wise.

$$
\begin{aligned}
(A_x, A_y, A_z) + (B_x, B_y, B_z) &= (A_x + B_x, A_y + B_y, A_z + B_z) \\
(A_x, A_y, A_z) - (B_x, B_y, B_z) &= (A_x - B_x, A_y - B_y, A_z - B_z) \\
c(A_x, A_y, A_z) &= (cA_x, cA_y, cA_z)
\end{aligned}
$$

4.7. THEOREM. ***Vector Equality****. Two vectors $\vec{A}$ and $\vec{B}$ are equal to each other if and only if their components along each unit vector (i.e. axis) are the same.*

PROOF. First note that a zero vector has zero components in all directions. Second, note that the statement $\vec{A} = \vec{B}$ means that $\vec{A} - \vec{B} = 0$. (Here 0 is a short-hand notation for the zero vector. $\vec{0} = 0\hat{i} + 0\hat{j} + 0\hat{k}$ in 3D, and $0\hat{i} + 0\hat{j}$ in 2D) □

4.8. NOTE. **Vector Equality (equivalent version)**

Two vecotrs are equal to each other, if and only if their magnitudes are equal and their directions are equal.

4.9. DEFINITION. **Magnitude of vector in Cartesian coordinates**

Given $\vec{A} = (A_x, A_y, A_z)$ in Cartesian coordinates (in 3D), its magnitude is given by $|\vec{A}| = \sqrt{A_x^2 + A_y^2 + A_z^2}$. In 2D, $\vec{A} = (A_x, A_y)$ and $|\vec{A}| = \sqrt{A_x^2 + A_y^2}$.

IV.2. Velocity and Acceleration Vectors

In a previous lecture, we actually dealt with velocity and acceleration vectors in 1D, but that was painless, since all we had to deal with were [signed] numbers, which are vectors in 1D (but not in higher dimensions). Now we have fancier objects – all these arrows – so we have to generalize our definitions.

4.10. DEFINITION. **Velocity vector**

Average velocity vector is defined as

$$\bar{\vec{v}} \equiv \frac{\Delta\vec{r}}{\Delta t} \tag{iv.1}$$

and the instantaneous velocity vector is its limiting form

$$\vec{v} \equiv \lim_{\Delta t\to 0} \frac{\Delta\vec{r}}{\Delta t} \equiv \frac{d\vec{r}}{dt} \equiv \dot{\vec{r}} \tag{iv.2}$$

Do not be scared by the derivative of a vector. Consider it as the derivative of each component. That is,

$$\dot{\vec{r}} = (\dot{x}, \dot{y}, \dot{z})$$

4.11. DEFINITION. **Speed**

The speed is defined as $|\vec{v}|$, and this definition is applicable in any dimension. Therefore, in 1D, it is $|v|$, where v is the velocity. In 2D, it is $\sqrt{v_x^2 + v_y^2}$, where $\vec{v} = (v_x, v_y)$. In 3D, it is $\sqrt{v_x^2 + v_y^2 + v_z^2}$. And similarly for higher dimensions, if necessary (?).

4.12. DEFINITION. **Acceleration vector**

Average acceleration vector is defined as

$$\bar{\vec{a}} \equiv \frac{\Delta\vec{v}}{\Delta t} \tag{iv.3}$$

and the instantaneous acceleration vector is its limiting form

$$\vec{a} \equiv \lim_{\Delta t\to 0} \frac{\Delta\vec{v}}{\Delta t} \equiv \frac{d\vec{v}}{dt} \equiv \dot{\vec{v}} \equiv \ddot{\vec{r}} \tag{iv.4}$$

4.13. NOTE. **Change is everywhere.**

In 1D, we are used to thinking about acceleration and deceleration. Indeed, on a nice straight freeway, that is all we do, i.e. pump gas to go faster and hit the brake to slow down. However, when on a freeway like I-17, things become more complicated we have to change the direction of the motion as we navigate through that meandering road. Even when speed does not change ($|\vec{v}|$ is kept constant) and car is changing its direction, there *is* an acceleration. It is just not speeding up or down. We really need to be aware of the vector nature of the position, and so the change of the position or the velocity can be in any direction.

IV.3. Reference Frame

Our experience tells us that there is a difference between local and global. For instance, we "know" that the earth is round, but locally we do not really feel it. We also "know" that the earth is moving very fast (how fast?), but on a quiet summer afternoon things cannot seem more still. And how about the earth going round the sun? And the sun...? A more practical example is when traveling by plane it takes less time to go west due to trade wind. Obviously we are in a seemingly never-ending hierarchy of environments, but intuitively we know that we can make sense of things as long as our view point is reasonably local. In physics, this "local viewpoint" is defined in terms of a local coordinate system, referred to as a "reference frame." As the above example makes it clear, two reference frames are in relative motion to each other, in general. For the plane in the wind example, say that the air in which the plane moves defines a reference frame A, and the earth be the reference frame B. If the plane is moving at velocity $\vec{v}_A$ relative to air, and the air is moving at $\vec{V}$, then the velocity of the plane relative to earth is

$$\vec{v}_B = \vec{v}_A + \vec{V} \tag{iv.5}$$

which summarizes the notion of "relative velocity."

IV.4. Constant Acceleration

OK, we have dealt with this problem in 1D, but what happens in 2D or 3D? We can most easily figure this out by considering each component of the vector, $\vec{a}$, $\vec{v}$, and $\vec{r}$. To be specific, consider 2D. We can write, for any acceleration,

$$\vec{r} = x\hat{i} + y\hat{j} = (x, y) \tag{iv.6}$$

$$\vec{v} = v_x\hat{i} + v_y\hat{j} = \dot{x}\hat{i} + \dot{y}\hat{j} = (v_x, v_y) = (\dot{x}, \dot{y}) \tag{iv.7}$$

$$\vec{a} = a_x\hat{i} + a_y\hat{j} = \dot{v}_x\hat{i} + \dot{v}_y\hat{j} = \ddot{x}\hat{i} + \ddot{y}\hat{j} = (a_x, a_y) = (\dot{v}_x, \dot{v}_y) = (\ddot{x}, \ddot{y}) \tag{iv.8}$$

Here, the notation $\ddot{}$ means the 2nd derivative. Note that, here, Definitions 4.10,4.12 have been used along with the fact that the time derivatives of $\hat{i}$ and $\hat{j}$ are zero. By constant acceleration, we mean that both a_x and a_y are constant. Recalling Theorem 4.7, these equations are separated into one set of equations for x components ($a_x = \dot{v}_x$, $v_x = \dot{x}$) and another set for y components ($a_y = \dot{v}_y$, $v_y = \dot{y}$). Solving each set of equations with their own initial conditions we obtain

$$\begin{aligned} v_x &= v_{x,0} + a_x t \\ x &= x_0 + v_{x,0}t + \frac{1}{2}a_x t^2 \end{aligned} \tag{iv.9}$$

and

$$\begin{aligned} v_y &= v_{y,0} + a_y t \\ y &= y_0 + v_{y,0}t + \frac{1}{2}a_y t^2 \end{aligned} \tag{iv.10}$$

Using $\vec{v}_0 = (v_{x,0}, v_{y,0})$ and $\vec{r}_0 = (x_0, y_0)$ we can summarize these two sets of equations of motions with more effectively as

$$\begin{aligned} \vec{v} &= \vec{v}_0 + \vec{a}t \\ \vec{r} &= \vec{r}_0 + \vec{v}_0 t + \frac{1}{2}\vec{a}t^2 \end{aligned} \tag{iv.11}$$

That is, these equations look the same as in 1D, *formally*, if we simply make the substitution $x\,(\text{or } y) \rightarrow \vec{r}$, $v \rightarrow \vec{v}$, and $a \rightarrow \vec{a}$, considering the explicit vector nature of these quantities in higher dimensions. In fact, Equation iv.11, although derived for 2D, remains valid for any dimension, including 1D. However, note that we rarely need to work in dimension higher than 2, since many problems in 3D are restricted to, or boil down to, motions in 2D subspace.

4.14. OBSERVATION. ***Divide and conquer***

To do a problem in 2D (or 3D), break it into separate problems for x,y(,z) components, respectively. Each problem is then a 1D problem, which can be solved readily.

