LECTURE III

Motion in 1D (cont.)

III.1. Integrating Equations of Motion

3.1. DEFINITION. **Initial conditions** [optional, if not familiar with integrals]

Suppose we know the function $a(t)$. How can we formally obtain $v(t)$ and $x(t)$? We can start by simply inverting Equation ii.2. We get

$$v(t) = v_0 + \int_{t_0}^{t} a(t')dt' \tag{iii.1}$$

Once we know $v(t)$, the process to go from $v(t)$ to $x(t)$ is just the same. Inverting Equation ii.1, we get

$$x(t) = x_0 + \int_{t_0}^{t} v(t')dt' \tag{iii.2}$$

Here, t_0 is a reference point in time (usually taken as 0), and we call it initial time. The values

$$v_0 = v(t = t_0) \tag{iii.3}$$

$$x_0 = x(t = t_0) \tag{iii.4}$$

are called *initial conditions*. Mathematically they are integration constants. We have two integrals to do to go from a to x, and thus we have two constants. Physically, of course, v_0 is the initial velocity and x_0 is the initial position.

3.2. FACT. ***Newtonian determinism*** [optional]

We just showed that if we knew $a(t)$, then we can simply integrate the equations $\dot{v} = a$ and $\dot{x} = v$ to solve for x. We will shortly consider an important case when a = constant, which definitely corresponds to this case. However, be warned that in many cases the problem is not that simple. As we will see later on in this course, in general, a is a *function* of x and v. This means that, in order to solve for x, v, we need to know them in the first place! Well, as bad as it sounds, this kind of situation happens all the time in physics and engineering. The situation is actually not really hopeless, either. In fact, using a slightly more sophisticated mathematical argument than presented in Definition 3.1, the following can be shown: the solution for x (and thus v and a) at any t can be obtained, as long as x_0 and v_0 are given. *Let me pause and emphasize* – if we know the position and the velocity of an object at one time [and of course the nature of forces in the problem], then we know the position and the velocity of that object at *all* times, from $t = -\infty$ to ∞. This is often called the "Newtonian determinism." Now, *how* do we obtain $x(t)$ in general? For simple problems we can obtain $x(t)$ using pencil and paper. We will deal with only such problems in this course. For more complex problems, computers can be used.

III.2. Constant Acceleration

The case of a constant acceleration is particularly simple and very important.

3.3. FACT. ***v(t) for constant a***

For constant acceleration a,

$$v(t) = v_0 + at \tag{iii.5}$$

where $v_0 \equiv v(t = 0)$.

PROOF. A direct way to see this is to use Eq. iii.1, with $t_0 = 0$, but here is another way, not involving integrals. For constant acceleration, $a = \bar{a}$. By Defs. 2.4,2.5, we have $a = \frac{\Delta v}{\Delta t} = \frac{v(t)-v(0)}{t-0}$. So, $v(t) = v(0) + at = v_0 + at$. □

3.4. FACT. ***x(t) for constant a***

For constant acceleration a,

$$x(t) = x_0 + v_0 t + \frac{1}{2}at^2 \tag{iii.6}$$

where $x_0 \equiv x(t = 0)$ *and* $v_0 \equiv v(t = 0)$.

PROOF. Using Eqs. iii.2 (with $t_0 = 0$) and iii.5, this answer is obtained immediately, but here is a slightly different way, not directly involving integrals. It is well-known from calculus that $\frac{d(t^n)}{dt} = nt^{n-1}$. Thus, $\frac{d(v_0 t)}{dt} = v_0$ and $\frac{d(\frac{1}{2}at^2)}{dt} = at$. Putting together we know that $x(t) = v_0 t + \frac{1}{2}at^2$ has the right property, namely $\dot{x}(t) = \frac{dx}{dt} = v(t)$, given in Eq. iii.5. But, wait! Any constant, when differentiated, is zero, and so how do we know that $x(t)$ does not have an additive constant? We don't, and it actually should have a constant. So $x(t) = const + v_0 t + \frac{1}{2}at^2$ is the most general (read "correct") solution. The *const* is $x(t = 0)$ and so we write it as x_0. □

3.5. EXAMPLE. **Free fall**

Let us say that you undergo a **free fall** (**motion** under the influence of **gravity alone**, i.e. no air resistance; **not** necessarily $v_0 = 0$) on a boardwalk ride with a downward acceleration $g \approx 9.8$ m/s^2 released with $v_0 = 0$ from height 19.6 m above the ground. (a) If suddenly the ride disappears except you, how long does it take for you to hit the ground? (ignore the size of the free fall object) (b) How fast will you be moving when you do?

SOLUTION. [1, Brute Force] (a) Take the coordinate system so that y represents the height. The y value increases as we go up, or, in other words, the y axis is pointing up. Initial conditions: $y_0 = 19.6$ m, and $v_0 = 0$ m/s. Equation of motion: $y = y_0 - \frac{1}{2}gt^2$ since $a = -g$. The minus sign means the acceleration is down, while the y axis is pointing up. [A more detailed way to see that a is negative is the following. Note that as an object falls the speed increases. However, acceleration is related to the increment of velocity, not of speed. The velocity v decreases, as its magnitude increases in the negative direction of the y axis. So, $\Delta v < 0$ for a positive Δt, which means $a = \lim_{\Delta t \to 0} \Delta v/\Delta t < 0$.] Plugging in values of y_0 and g, we get $y = 19.6\,\text{m} - \frac{1}{2} \cdot 9.8\,\frac{\text{m}}{\text{s}^2} \cdot t^2$. We need to figure out the value of t when $y = 0$. Setting $y = 0$ (and dividing the whole equation by the unit m), we get $19.6 - 4.9\frac{1}{\text{s}^2}t^2 = 0$. So, $t^2 = \frac{19.6}{4.9}\text{s}^2 = 4.0$ s^2. Thus, $t = 2.0$ s. (b) $v = -gt$ and so when $t =$2.0 s, $v = -9.8\ \frac{\text{m}}{\text{s}^2} \times 2.0$ s $= -2.0 \times 10^1$ m/s. Since the question asks "how fast," we can just quote the speed and say "at 2.0×10^1 m/s." [Order of magnitude check: how many mph is that?] □

OK, this solution is good, but I actually do *not* recommend doing a problem this way. What would be a better way?

SOLUTION. [2, Less Brute Force] (a) $y = y_0 - \frac{1}{2}gt^2$ so when $y = 0$ we have $t = \sqrt{2y_0/g}$. [Dimension check: $\sqrt{L/(L/T^2)} = T$, good!] Plugging in numbers for y_0 and g *now*, we get $t = \sqrt{2 \cdot 19.6\,\text{m} \ /(9.8 \ \text{m/s}^2)} = 2.0$ s. (b) No change. **Lesson**: **Work out the algebra with symbols rather than numbers. Then, at the end, plug in numbers.** □

Let me finish by offering another solution.

SOLUTION. [3, Optional] Define the y axis so that it is pointing *down*, and its origin is at the highest point of the free fall motion. In this case the acceleration is positive. Then, $y = \frac{1}{2}gt^2$ and $v = gt$. So, $t = \sqrt{2y/g}$. When you reach the bottom $y = 19.6$ m. The rest of this solution proceeds just like in the second solution ($t = \sqrt{...} = 2.0$ s). □

3.6. EXAMPLE. **Free fall, abstract**

An object is undergoing a free fall with a downward acceleration g released with $v_0 = 0$ from height h above the ground. **(As in the previous problem we actually ignore the size of the object. This will be implicitly the case for all problems that we do in this course, unless of course the problem explicitly takes the size into consideration.)** What is the time t it takes for the object to hit the ground? What is the speed at which it hits the ground? Looking at your answer, find the answer to the following question. Suppose you want to double that speed. How much do you need to scale up the height?

SOLUTION. The first question is already answered in the previous example. $t_G = \sqrt{2h/g}$, where I use the subscript G to mean the time it touches the ground, not just any time. How about the velocity? In this case, $v = -gt$, and so all we need to do is to plug in t_G to get v_G. That is $v_G = -\sqrt{2gh}$. Well, the question actually asks the speed, so the answer is $|v_G| = \sqrt{2gh}$. Now, if we want to double $|v_G|$ then it is clear that h should be quadrupled. **Lesson: working with symbols is essential for answering questions regarding scaling behaviors or dependence on parameters.** □

3.7. EXAMPLE. **Stopping distance**

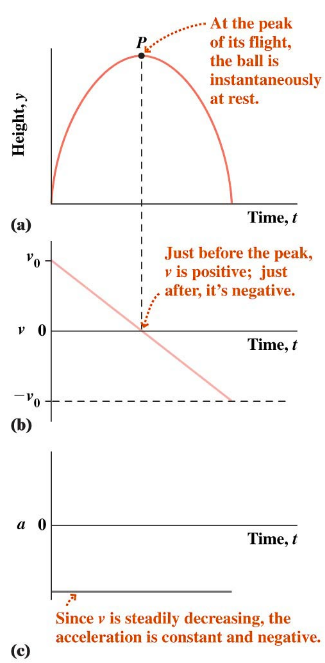
You are driving at speed v_0. Suddenly the car in front of you stops. You hit the brake, and as the result the speed of your car decreases. Let us assume that the speed decreases at a constant rate b ($b > 0$). What is the time t_s it takes for the car to stop, in terms of v_0, b? What is the stopping distance d_s that the car travels from the moment you hit the brake to the moment the car stops? Using $b \approx 0.7g$ (≈ 7 m/s^2), make a table of t_s (second) and d_s (meters) for $v_0 = 20$, 40, 60, 80 mph.

SOLUTION. This problem is almost the same as the free fall problem. Let us use x for our axis, since this is a horizontal motion. (It really does not matter what symbol we use. We just have to use it consistently in a given problem.) Define the time you hit the brake as $t = 0$. And the position of the car at $t = 0$ as $x = 0$. If we define the x axis to point in the direction in which the car is moving, then the acceleration in this case is $-b$, since the velocity is decreasing in this coordinate system. So, $x = v_0t - \frac{1}{2}bt^2$ and $v = v_0 - bt$. The car stops when $v = 0$, and so $0 = v_0 - bt_s$. $t_s = v_0/b$. The stopping distance is then $d_s = v_0t_s - \frac{1}{2}bt_s^2 = v_0\frac{v_0}{b} - \frac{1}{2}b(\frac{v_0}{b})^2 = v_0^2/(2b)$. This is the answer. Note that if v_0 doubles then d_s quadruples! This is the reason to not drive too fast. Also, did you notice how nicely the two terms v_0t_s and $\frac{1}{2}bt^2$ end up being same expressions except numerical factors ($\frac{v_0^2}{b}$ and $-\frac{1}{2}\frac{v_0^2}{b}$)? This is not a coincidence at all. The underlying reason is that two add-able terms must have the same dimension (cf. Fact 1.7). You will see many examples like this when you do problems. Generally, if your answer is too complicated (for instance if the two terms here did not add up to a nice single expression), it is a good idea to stop and double check your previous steps. Finally, here is the table (answers for t_s and d_s have only one significant figure, due to b having only one significant figure, as given in this example $b \approx 7$ m/s^2). Consider that on a slippery surface, b can be reduced by as much as a factor of 2, which will scale up t_s and d_s by that factor! □

v_0 (mph)	20	40	60	80
v_0 (m/s)	8.9	18	27	36
$t_s = v_0/b$ (s)	1	3	4	5
$d_s = v_0^2/(2b)$ (m)	6	20	50	90

3.8. EXAMPLE. **Vertically tossed ball**. Imagine a ball thrown up vertically from the ground (zero height). This is also a 1D problem. Let us use y as the position variable representing the height. The acceleration in this case is due to gravity, and again we can approximate it as $-g$, where g is a positive number and the minus sign in front of g means that the acceleration is down, i.e. the negative direction of the y axis. In this problem, the initial conditions are: $y(t=0) = 0$ and $v(t=0) = \dot{y}(t=0) = v_0$ (the speed at which the ball is tossed upward). Applying Equations iii.5,iii.6, we get

$$
\begin{aligned}
v(t) &= v_0 - gt \\
y(t) &= v_0 t - \frac{1}{2}gt^2
\end{aligned}
$$



These are some of typical questions that it is important to know how to answer.

(a) What is the value of v when $y = h$, where $h \equiv$ the maximum height? *Ans:* 0.

(b) What is the relationship between h and v_0? *Ans:* $v_0 = \sqrt{2gh}$. *Why:* When $y = h$, $v = 0$. So, $0 = v_0 - gt$ and $h = v_0 t - \frac{1}{2}gt^2$. Eliminating time t, $h = v_0 \frac{v_0}{g} - \frac{1}{2}g\frac{v_0^2}{g^2} = \frac{1}{2}\frac{v_0^2}{g}$. So, $v_0 = \sqrt{2gh}$.

(b) When does the speed become maximum? *Ans:* $t = 0$ and $t = 2\sqrt{2h/g}$. *Why:* The speed is maximum at $y = 0$, which means $v_0 t - \frac{1}{2}gt^2 = 0$. One solution is $t = 0$ and the other is $t = 2v_0/g = 2\sqrt{2h/g}$ using the result of (b).

3.9. EXERCISE. Get answers for (b,c) above by breaking down the motion into two, and making use of the answers for the free fall problem considered above. Note that the 2nd half of this motion is just a free fall, while the 1st half is a free-fall "played backwards."

3.10. SUMMARY. **Key results for 1D motion with constant acceleration**

The following table from the text summarizes key results for a constant acceleration motion.

TABLE 2.1 Equations of Motion for Constant Acceleration

Equation	Contains	Number
$v = v_0 + at$	v, a, t; no x	2.7
$x = x_0 + \frac{1}{2}(v_0 + v)t$	x, v, t; no a	2.9
$x = x_0 + v_0 t + \frac{1}{2}at^2$	x, a, t; no v	2.10
$v^2 = v_0^2 + 2a(x - x_0)$	x, v, a; no t	2.11

Of these four equations, the 2nd and the 4th equations are new here, and need derivations. The 2nd equation:

$$\begin{aligned} \frac{1}{2}(v_0 + v)t &= \frac{1}{2}(v_0 + v_0 + at)t \\ &= v_0 t + \frac{1}{2}at^2 \\ &= x - x_0 \end{aligned}$$

The 4th equation:

$$\begin{aligned} v^2 - v_0^2 &= (v_0 + at)^2 - v_0^2 \\ &= 2v_0 at + a^2t^2 \\ &= 2a(v_0 t + \frac{1}{2}at^2) \\ &= 2a(x - x_0) \end{aligned}$$

Your are not required to intentionally memorize these two extra equations. However, if their meanings are clear, then they may be very easy to recall even if you do not care to memorize. $\frac{1}{2}(v_0 + v)t = x - x_0$ is the statement about the average velocity. In the case of a constant acceleration, the velocity changes linearly as a function of time, and thus the average velocity is accurately given by the average of the initial and the final velocities, i.e. $(v_0 + v)/2$. Equating this to the definition of the average velocity $\Delta x/\Delta t = (x - x_0)/t$, we get the desired result. The last one, $v^2 - v_0^2 = 2a(x - x_0)$ is the statement about the energy conservation, which we will cover in a later lecture. Until then, I will not require that you can pull out this equation from top of your head.

3.11. OBSERVATION. ***Instantaneous wins.***

Instantaneous velocity/acceleration (and position, for that matter) is generally a more fundamental quantity than average velocity/acceleration/position, and so if the "instantaneous" or "average" nature is un-specified, then it should be interpreted as "instantaneous."