APPENDIX D

Single variable calculus in a nutshell

In this appendix, key results of single variable calculus are summarized, from the physicist point of view. Note that in standard calculus literature x tends to be the notation for the single variable. Here, we use t instead, as the time-derivative is important in this course. However, keep in mind that the single variable can be (time or position or any other physical variable) when you apply the results of this appendix to physical situations.

How to read this appendix: Read those items whose title starts with $[\star]$. They are more essential for this course.

D.1. Differentiation

D.1. DEFINITION. $[\star]$ $\Delta t \to 0$, dt. **Infinitesimal is close to, but *not* equal to, zero.**

In the calculus lingo, Δt is a smallish increment in t, often denoted as h. In this note, I will use Δt and h interchangeably. The meaning of $\Delta t \to 0$ is "make Δt very very small, but *not* zero." How small? *The* answer is "much smaller than the experimental resolution of a given problem." This is the notion of an "infinitesimal" in mathematics. Mathematicians often use symbols such as ε, δ for infinitesimal quantities. More importantly for this course, for a physical variable Q, e.g. t (time), x (position), v (velocity), E, (energy) or V (volume), its infinitesimal is denoted as dQ, e.g. dt, dx, dv, dE, or dV[1].

D.2. DEFINITION. $[\star]$ $h \to 0$, $h \to 0^+$, $h \to 0^-$

A variable can approach zero from the positive axis (as in $h \to 0^+$), or approach zero from the negative side (as in $h \to 0^-$) or approach zero from either side (as in $h \to 0$).

D.3. DEFINITION. $[\star]$ **Differentiable function and its first derivative,** $\frac{df}{dt}$ **or** $\frac{d}{dt}f$ **or** $f'(t)$

For a given function $f(t)$, consider $\Delta t = h$ (which may be negative), and $\Delta f = f(t+h) - f(t)$. If

$$\lim_{h \to 0} \frac{\Delta f}{\Delta t}$$

converges, (namely, the limit value remains finite and unique for both limits $h \to 0^+$ or $h \to 0^-$) then we say that the function is differentiable and the *first derivative*, or simply "*the derivative*," of $f(t)$ is defined as that limit value

$$\frac{df}{dt} \equiv \frac{d}{dt}f \equiv f'(t) \equiv \lim_{\Delta t \to 0} \frac{\Delta f}{\Delta t} \equiv \lim_{h \to 0} \frac{f(t+h) - f(t)}{h}$$

[1] Symbols such as δx, δv, δE, or δV are also found in the science literature to mean a similar thing – a virtual infinitesimal.

D.4. NOTE. **Left and right derivatives**

When can $\lim_{h\to 0}\frac{\Delta f}{\Delta t}$ be *not* well-defined? Consider $f(t)=|t|$ and let us try to take the derivative at $t=0$. If we take h to be positive, then

$$\lim_{h\to 0^+}\frac{\Delta f}{\Delta t}\bigg|_{t=0}=\lim_{h\to 0^+}\frac{f(h)-f(0)}{h}=\lim_{h\to 0^+}\frac{h}{h}=1$$

while

$$\lim_{h\to 0^-}\frac{\Delta f}{\Delta t}\bigg|_{t=0}=\lim_{h\to 0^-}\frac{f(h)-f(0)}{h}=\lim_{h\to 0^+}\frac{-h}{h}=-1$$

Here, $|_{t=0}$ means "evaluated at $t=0$". As you can see, depending on whether h approaches 0 from right or left, the limit value is different! In this case, $\lim_{h\to 0}\frac{\Delta f}{\Delta t}$ is not well defined at $t=0$, since it is ambiguous. However, people can/do still talk separately about the *left derivative* ($h\to 0^-$) and the *right derivative* ($h\to 0^+$). Such strange functions do exist in physics (and engineering and science). In fact, such functions tend to play particularly important roles and give particularly useful information!

D.5. EXAMPLE. [$\star$] Prove that $(t^n)'=nt^{n-1}$ where $n\geq 0$.

PROOF. We need to evaluate $\lim_{h\to 0}\frac{(t+h)^n-t^n}{h}$. Note that $(t+h)^n=(t+h)(t+h)...(t+h)$ $=t^n+nt^{n-1}h+...$, where the omitted terms, if they exist, are higher powers of h. Thus $\frac{(t+h)^n-x^n}{h}=\frac{nt^{n-1}h+...}{h}=nt^{n-1}+....$ As $h\to 0$, the omitted terms will converge to 0, as they contain at least one power of h. This completes the proof for $n\geq 1$. For $n=0$, $(t+h)^n=1$ and $t^n=1$ and so $\lim_{h\to 0}\frac{(t+h)^n-t^n}{h}=\lim_{h\to 0}\frac{0}{h}=0$ (note that h becomes infinitesimal when $h\to 0$; i.e. it is never zero). On the other hand, nt^{n-1} is also 0 if $n=0$. □

D.6. NOTE. It turns out that the above relation can be generalized to any exponent (negative integers or even fractions), as long as the functions on the LHS and the RHS are well-defined. See theorem D.12 below. You are assumed to be able to use this differentiation rule for any exponent in doing problems in this course.

D.7. DEFINITION. **The n-th derivative, $\frac{d^nf}{dx^n}$ or $\left(\frac{d}{dx}\right)^n f$ or $f^{(n)}(t)$, $n=0,1,2,...$**

The n-th derivative, $n\geq 1$, is defined as the first derivative of the $(n-1)$-th derivative $f^{(n-1)}(t)$. The definition is made complete for all non-negative integers $n\geq 0$ by defining $f^{(0)}(t)\equiv f(t)$.

D.8. EXAMPLE. Find $f^{(n)}(t)$ $(n=1,2,...)$ for (a) $f(t)=at^2+b$ and (b) $f(t)=|t^2|$.

PROOF. (a) $f^{(1)}(t)=2at$. $f^{(2)}(t)=2a$. $f^{(n)}(t)=0$ if $n>2$. (b) In this case, $f(t)=t^2$ for $t\geq 0$, and $f(t)=-t^2$ for $t\leq 0$. Thus, $f'(t)=2t$ for $t\geq 0$, and $-2t$ for $t\leq 0$. This can be nicely summarized as $f^{(1)}(t)=2|t|$. Differentiating $f^{(1)}(t)$, we have $f^{(2)}(t)=2$ for $t>0$ and -2 for $t<0$. The reason that $t=0$ is left out here is because $f^{(2)}(t)$ is not well-defined, 2 and -2 being different. Continuing, $f^{(n)}(t)=0$ for $n>2$ and $t\neq 0$. In this example, $f^{(n)}(t=0)$ is *not well-defined* for $n>1$ at $t=0$. □

D.9. DEFINITION. **Analytic function, infinitely differentiable**

A function $f(t)$ is said to be analytic in a set ("domain") of t, if $f^{(n)}(t)$ is well-defined for any $n=1,2,...$ and for *all* values t in that set. So, in the above example, at^2+b is analytic, while $|t^2|$ is not, if the domain includes $t=0$.

D.10. OBSERVATION. ***Analytic functions are smooth. Non-analytic functions are not.***

PROOF. Graph the functions of the above example to get a feel for this statement. □

D.11. NOTE. [⋆] **Leibniz notation ($\frac{d^n f}{dt^n}$, $\left(\frac{d}{dt}\right)^n f$) and Newton notation ($f'$, f'', ..., $\dot{f}$, $\ddot{f}$, ...)**

Newton and Leibniz are the two independent inventors of calculus. The Leibniz notation is powerful, while the Newton notation is compact. The power of the **Leibniz notation** lies in the fact that "d" can be regarded as the infinitesimal "operator" (see Definition D.1)[2]. The **Newton notation** has two kinds, the "prime" kind and the "dot" kind. In either case, the number of symbols used, prime or dot, corresponds to n of the Definition D.7. Thus, $f''(t) = f^{(2)}(t)$, for example. By convention, the dot notation is used *only when the argument of the function is time t*. So, for $f(x)$ the dot notation is *not* used, while for $v(t)$, the dot notation can be used, as in $\dot{v}(t)$. Unfortunately, the prime notation is commonly used simply to mean "other" values, as in "one car is moving at speed v while another care is moving at speed v'." For this reason, ***Newton's prime notation will never be used in my lecture note except in this appendix***.

D.12. THEOREM. [⋆] ***Very important basic rules of differentiation***

(a) $(1)' = 0$ (b) $(t)' = 1$ (c) $(fg)' = f'g + fg'$ (d) $(f/g)' = (f'g - fg')/g^2$ (d) $(Af + Bg)' = Af' + Bg'$ (e) $(At^\alpha)' = A\alpha t^{\alpha-1}$ for *any* constant α, not just an integer α (as long as the functions on both sides of this equation are well-defined) (f) $(f(g(t)))' = f'(g(t))g'(t)$ or $\frac{df(g(t))}{dt} = \frac{df(g)}{dg}\frac{dg}{dt}$ (chain rule for a composite function). Here, f and g are functions of t (except in (f), f is a function of $g(t)$) and A and B are constants. Note that in (d) g^2 means simply the square of g (not to be confused with $g^{(2)}$, which would mean the second derivative by Definition D.7).

PROOF. Left to readers. All rules follow from Definition D.3. *The importance of the ability to comfortably apply any combination of these rules for solving real problems cannot be over-emphasized*. Note that the chain rule can be understood as an algebraic cancellation of the infinitesimal dg in the Leibniz notation $\frac{df}{dg}\frac{dg}{dx} = \frac{df}{dx}$. □

D.13. EXAMPLE. Find $f'(t)$ when $f(t)$ is (1) 50 (2) $-10t$ (3) t^2 (4) $1/t$ (5) $t + 1/t$ (6) $\sqrt{t}$ (7) $\sqrt{t^2+1}$.

PROOF. (1) By (a,d) above, $(50)' = 50 \cdot 0 = 0$. (2) By (b,d), $(-10t)' = -10 \cdot 1 = -10$. (3) This can be simply solved by applying (e), $(t^2)' = 2t$, but, for demonstration, let us also do it by applying (b,c): $(t^2)' = (tt)' = 1 \cdot t + t \cdot 1 = 2t$. (4) Again, this can be solved by applying (e), $(t^{-1})' = (-1) \cdot t^{-2}$. but, for demonstration, let us also do it by applying (b,d): $(1/t)^{-1} = (0 \cdot t - 1 \cdot 1)/t^2 = -1/t^2$. (5) Applying (b,d), $(t + 1/t)' = 1 + (1/t)'$. Using the result that we just obtained in (4), this is equal to $1 - 1/t^2$. (6) Using (e), $(\sqrt{t})' = (t^{1/2})' = \frac{1}{2}t^{-1/2} = 1/(2\sqrt{t})$. (7) We need to use the chain rule (f) for this one. Take $f(g) = \sqrt{g}$ and $g(t) = t^2 + 1$. Then $\sqrt{t^2+1} = f(g(t))$ is a composite function, and the meaning of the chain rule is to differentiate f w.r.t. its argument (g) and

[2]Keep in mind that dt should be regarded as meaning one quantity, *not* (!) the product of two symbols d and t. If you keep this in mind, then you will appreciate, for example, that d in df/dt is not just a symbol that can be canceled out (!). The same comment applies for the increment "operator" Δ in $\Delta f/\Delta t$.

then multiply it by g'. We know $f'(g) = 1/(2\sqrt{g})$ from (6). So, the answer here is $\frac{1}{2\sqrt{g}} \cdot g'(t) = \frac{1}{2\sqrt{t^2+1}} \cdot 2t = t/\sqrt{t^2+1}$. □

D.14. NOTE. [⋆] **Meanings of $f'(t)$**

$f'(t)$ represents the tangential slope at t for a given graph $f(t)$.

D.15. EXAMPLE. [⋆] **Derivatives of some useful elementary functions**

Here a, b, α are constants.

- $\frac{d}{dt}[(at+b)^\alpha] = \alpha(at+b)^{\alpha-1}$ assuming both LHS and RHS are well-defined for non-integer α
- $\frac{d}{dt}\sin(at+b) = a\cos(at+b)$
- $\frac{d}{dt}\cos(at+b) = -a\sin(at+b)$
- $\frac{d}{dt}e^{at+b} = ae^{at+b}$ (also note that e^t and $\exp(t)$ are two notations for the same function)
- $\frac{d}{dt}\ln(at+b) = \frac{a}{at+b}$ assuming $(at+b > 0)$

D.2. Integration

D.16. DEFINITION. [⋆] **Definite integral from a to b**

$$\int_a^b f(t)dt \equiv \lim_{N\to\infty} \sum_{n=0}^{N} f(t_n)h \tag{D.1}$$

where $t_n = a + nh$, $h = (b-a)/(N+1)$, and N is a positive integer. Note that in this definition it is *not* necessary that $a < b$, and accordingly h can have any sign. We call the function $f(t)$, which is within the integration symbol ($\int$), the *integrand*. The positions of $f(t)$ and dt are interchangeable: $\int_a^b dt f(t) \equiv \int_a^b f(t)dt$. a (b) is called the lower (upper) limit of the integration.

D.17. OBSERVATION. ***Infinitesimal, again***

As in the differentiation, the integral involves the infinitesimal dt. Where does this come from? It comes from h, which is the increment of the variable t, since $t_n = a + nh$ on the right hand side (RHS). Thus, h becomes the infinitesimal dt when $N \to \infty$.

D.18. DEFINITION. **Dummy variable**

This trivial concept is *very* important to understand, because if you do not understand it you can get lost quite often. In an expression, a variable whose symbol can be replaced by another symbol without affecting the outcome of the expression is called a dummy variable.

D.19. EXAMPLE. **Summation and dummy variable**

In a sum like $\sum_{n=0}^{M} 4n^2$, n in the summed expression ("summand") $4n^2$ is a dummy variable. On the contraray, M is not a dummy variable. If you need an explanation, read on. The sum means $4(0^2 + 1^2 + ... + M^2)$. It does not matter what symbol we use to conveniently denote the series of integers 1, 2, ... M. Conventionally, we use symbols like i, j, k, l, m, n for integers. So, we might use expressions like $\sum_{m=0}^{M} 4m^2$, which means exactly the same quantity. In conclusion, m or n used in the summand are dummy variables. This can be summarized by saying that "***any summation variable is a dummy variable***." The next fact is a direct consequence of this fact.

D.20. FACT. [⋆] ***Any integration variable in a definite integral is a dummy variable, namely*** $\int_a^b f(t)dt = \int_a^b f(u)du$

PROOF. If you need an explanation for this, no problem, read on. The meaning of Equation D.1 is that for large N, the sum $\sum_{n=0}^{N} f(t_n)h$ becomes incredibly close to, namely converges to, a well-defined value, which we define as $\int_a^b f(t)dt$. Let me define the sum as $S_N(a,b)$. Since $t_n = a + nh$, we can write $S_N(a,b) = \sum_{n=0}^{N} f(a+nh)h$. At this point notice that there is nothing special about that symbol, t_n. We might as well call $a + nh$ something else like y_n, and it cannot, in any way, change the fact that $S_N(a,b) = \sum_{n=0}^{N} f(a+nh)h$, and that the sum S depends only on a, b, N and the function f and nothing else (like t_n or y_n). So, $S_N(a,b) = \sum_{n=0}^{N} f(y_n)h$. In this notation, the increment of y_n is h and thus the limiting value of this sum should be written as $\int_a^b f(y)dy$. Thus, $\int_a^b f(t)dt = \int_a^b f(y)dy$. □

D.21. THEOREM. [⋆] ***Very important basic properties of definite integral***

(a) $\int_a^b f(t)dt = -\int_b^a f(t)dt$ (b) $\int_a^a f(t)dt = 0$ (c) If $f(t) = C$ (constant), then $\int_a^b f(t)dt = (b-a)C$. (d) $\int_a^b [Af(t) + Bg(t)]dt = A\int_a^b f(t)dt + B\int_a^b g(t)dt$ for any constants A, B. (e) $\int_a^b f(t)dt = \int_a^c f(t)dt + \int_c^b f(t)dt$ for *any* number c, regardless of whether c inside or outside the range $[a,b]$. (f) If $a \le b$ and $f(t) \ge 0$, then $\int_a^b f(t)dt \ge 0$. (g) If $a \le b$ and $f(t) \le g(t)$, then $\int_a^b f(t)dt \le \int_a^b g(t)dt$.

PROOF. Left to readers. These properties follow directly from Definition D.16. Note that (g) is a direct consequence of (d,f). □

D.22. NOTE. [⋆] **Meaning of $\int_a^b f(t)dt$, assuming $a < b$**

(i) If $f(t) \ge 0$ for $a < t < b$, then $\int_a^b f(t)dt$ gives the area below the graph $f(t)$ (bounded by the t axis and $t = a$ and $t = b$). (ii) If $f(t) \le 0$ for $a < t < b$, then $\int_a^b f(t)dt$ gives the negative of the area above the graph $f(t)$ (bounded by the t axis and $t = a$ and $t = b$). (iii) If $f(t)$ changes sign between $t = a$ and $t = b$? First, break down the range $[a,b]$ into sub-ranges in each of which the function $f(t)$ has only one sign. Applying (c) of Theorem D.21 as many times as necessary, the integral $\int_a^b f(t)dt$ can be split into integrals over those sub-ranges. Each integral over a sub-range can then be interpreted according to (i) or (ii). Lastly, if $a > b$, then $\int_a^b f dt = -\int_b^a f dt$ can be used to make use of the results of this note.

D.23. THEOREM. ***Mean value theorem (integral version)***

For a continuous function $f(t)$ and $a \le b$, $\int_a^b f(t)dt = f(c)(b-a)$ for some c in the range $[a,b]$.

PROOF. [optional] This is one of the major theorems in calculus. Consider the case $a = b$, first. In this case the integral is 0, and the right hand side is satisfied trivially by taking $c = a$. Now consider the case $a < b$. Consider a function $g(t) = C$ where the constant $C = f(t_{max})$, where t_{max} is the value of t at which $f(t)$ is (the) maximum in the interval $[a,b]$. Applying theorem D.21(g), $\int_a^b f(t)dt \le \int_a^b g(t)dt = \int_a^b Cdt$. Applying therm D.21(c) to the last expression, $\int_a^b Cdt = (b-a)C$. Thus, we have $\int_a^b f(t)dt \le f(t_{max})(b-a)$. Similarly, one can prove $f(t_{min})(b-a) \le \int_a^b f(t)dt$. In

summary, $f(t_{min})(b-a) \le \int_a^b f(t)dt \le f(t_{max})(b-a)$. Dividing the entire inequality by $b-a$, which is a positive number, we get $f(t_{min}) \le \frac{1}{b-a}\int_a^b f(t)dt \le f(t_{max})$. Now, the continuity of $f(t)$ means that for any number y_{mid} for which $f(t_{min}) \le y_{mid} \le f(t_{max})$ we can find c for which $a \le c \le b$ and $f(c) = y_{mid}$. We may have more than one choice for c. The point is that at least one value of c is guaranteed to exist by the continuity. Thus, this proof is completed by identifying $\frac{1}{b-a}\int_a^b f(t)dt$ as y_{mid}. □

D.24. DEFINITION. [⋆] **Indefinite integral, integrability and integration constant**
For $f(t)$, its indefinite integral $F(t)$ is defined as

$$F(t) \equiv \int_a^t f(y)dy \equiv \int^t f(y)dy \equiv \int f(t)dt \tag{D.2}$$

where a is an *arbitrary* constant (and thus omitted in the 2nd form). Notice that in this equation y is a dummy variable, but t is *not*.

A function $f(t)$ is said to be *integrable* within a certain range of t, if its indefinite integral $F(t)$ exists for any value of t within that t-range. *Integrable functions do not need to be continuous, although continuous functions are always integrable.*

Note that a is arbitrary in the sense that any choice of a is fine as long as $f(t)$ is integrable in the range $[a,t]$. Say we have two choices a_1 (e.g. 0) and a_2 (e.g. 0.5). The difference of the two functions $F(t)$ corresponding to the two choices of a is $\int_{a_1}^t f(y)dy - \int_{a_2}^t f(y)dy$. Using theorem D.21(a) for the second term, this becomes $\int_{a_1}^t f(y)dy + \int_t^{a_2} f(y)dy$. Using theorem D.21(e), this becomes $\int_{a_1}^{a_2} f(y)dy$, which has no t dependence. What this means is that indefinite integral $F(t)$ is not unique. For a given $F(t)$, $F_2(t) = F(t) + C$, where C is a constant, is also a valid indefinite integral of $f(t)$. The arbitrary constant C is called the *integration constant* or the *constant of integration*. Choosing a fixed value for a in Equation D.2 is equivalent to choosing a fixed value of C.

Lastly, note that the last expression $\int f(t)dt$ is by far the most common notation for the indefinite integral. In this note, however, I will use $\int^t f(y)dy$ as this notation separates the variable and the dummy variable with much more clarity. Of course, in some sense all arguments of any function are dummy, and $\int f(t)dt$ is OK, assuming that the reader knows that at any given moment which variable is dummy and which is not.

D.25. OBSERVATION. ***Human nature***

People very often use just "integral" to mean either "indefinite integral" or "definite integral," as long as the context makes it clear what they mean.

D.26. THEOREM. [⋆] ***For a continuous function*** $f(t)$***, its indefinite integral*** $F(t)$ ***is differentiable, and***

$$\frac{dF(t)}{dt} = f(t) \tag{D.3}$$

PROOF. Consider $F(t+h) = \int_a^{t+h} f(y)dy$. Using theorem D.21(e), $F(t+h) = \int_a^t f(y)dy + \int_t^{t+h} f(y)dy = F(t) + \int_t^{t+h} f(y)dy$. Since $f(t)$ is a continuous function, theorem D.23 is applicable for $\int_t^{t+h} f(y)dy$, and we conclude $F(t+h) - F(t) = \int_t^{t+h} f(y)dy = f(c)h$ where $t \le c \le t+h$. Consider now the derivative of $F(t)$, $F'(t) = \lim_{h\to 0}(F(t+h) - F(t))/h = \lim_{h\to 0}(f(c)h)/h = \lim_{h\to 0} f(c)$. The proof is complete if one notes $c \to t$ as $h \to 0$. □

D.27. NOTE. [⋆] **Equations of motion and initial conditions**

In physics, we encounter many types of equations of motion. The simplest example is $\dot{v}(t) = a(t)$: say $a(t)$ is some known acceleration and then we are supposed to determine $v(t)$. This is exactly in the *form* of Equation D.3, and thus $v(t)$ is an indefinite integral of $a(t)$. Knowing $a(t)$ is not enough to know $v(t)$ unambiguously, because of the arbitrariness of the integration constant. However, if we know the velocity value at any one point of time, for example $v(t = 0)$, then that can remove this ambiguity. This way, an *integration constant* in calculus is related to an *initial condition* in physics.

D.28. NOTE. **Integration makes it nicer, while differentiation makes it worse.**

Note that in general the integral function is nicer than the integrand function: if $f(t)$ is a continuous function, then $F(t) = \int^t f(y)dy$ is differentiable (and so it is more than continuous), and then the successive integration process can go on indefinitely. Conversely, $f'(t)$ is in general worse than $f(t)$, and, if one keeps differentiating a function $f(t)$, then that process is not guaranteed to go on indefinitely (for example consider repeatedly differentiating $f(t) = |t^2|$ at $t = 0$ as in Example D.8). However, there is a special class of functions, analytic functions (Definition D.9), which can be differentiated as many times as one likes. This fact can be summarized by saying that "the integral or the derivative of an analytic function is also an analytic function."

D.29. NOTE. **Integration of a discontinuous function**

Say $f(t) = 0$ for $t < 0$ and 1 for $t > 0$. Define $f(0) = 1/2$. This completes the definition of the famous "step function" (or the Heaviside function). Obviously, it is a "bad" function, in the sense that it is discontinuous at $t = 0$. In science and engineering, it turns out that this is an extremely useful function. OK, let us integrate it to get $F(t)$. Let us define it as $F(t) = \int_0^t f(y)dy$. From definition D.16, it is easy to figure out that $F(t) = 0$ for $t \leq 0$ and $F(t) = t$ for $t \geq 0$. So this example shows that a discontinuous function can become continuous on integration – this is an example of "integration makes it nicer." Now, let us differentiate $F(t)$. The question is – do we go back to $f(t)$? The answer is – not quite. This is because, at $t = 0$, $F'(t)$ is not well-defined. So, even though the integral is the inverse of the derivative in the sense of Eq. D.3, this is only in so far as $F'(t)$ is well defined, which may not be the case for a non-continuous $f(t)$.

D.30. THEOREM. [⋆] ***Definite integral in terms of indefinite integral***

If $F(t)$ is an indefinite integral of $f(t)$, then $\int_a^b f(t)dt = F(b) - F(a)$. This result is compactly written as $F(t)|_a^b$.

PROOF. This follows from theorem D.21(e,a) and Eq. D.2. Here is how it goes. $\int_a^b f(t)dt = \int_a^c f(t)dt + \int_c^b f(t)dt$ (c is an arbitrary constant) $= -\int_c^a f(t)dt + \int_c^b f(t)dt$ $= -F(a) + F(b)$, where in the last step $F(t) = \int_c^t f(y)dy$ is used. □

D.31. THEOREM. ***Mean value theorem (differential version)***

For a differentiable function $f(t)$, consider a t-range $[a, b]$. Then there exists a c in the range (a, b) for which $f'(c) = (f(b) - f(a))/(b - a)$.

PROOF. Actually this is an immediate consequence of theorem D.23 and the theorem that we just proved. Just take $f(t)$ here to be $F(t)$ in theorem D.23. □

D.32. THEOREM. [⋆] ***Very important basic properties of indefinite integral***

All properties of the definite integrals in theorem D.21 are applicable to indefinite integrals if the upper limit, if different from the lower limit, is taken to be a variable instead of a constant. In addition, the following properties are important.

(a) If $f(t) = t^\alpha$, then $\int^t f(u)du = \frac{1}{\alpha+1}t^{\alpha+1} + C$. (b) $\int_a^b F(u)g(u)du = F(u)G(u)\big|_a^b - \int_a^b f(u)G(u)du$, where $F(u)$ ($G(u)$) is an indefinite integral of $f(u)$ ($g(u)$). (c) $\int_a^b f(g(t))dt = \int_{g(a)}^{g(b)} f(y)\frac{dy}{g'(t(y))}$ (chain rule).

PROOF. These properties follow from the fact that the integral is the inverse of the derivative. For (a) use theorem D.12(e), and for (b) use theorem D.12(c), and lastly for (c) use D.12(f). Familiarity with these properties would be very beneficial, perhaps essential, for your success physics courses. □

D.33. EXAMPLE. (a) $\int^t (u+1)du$ =? (b) $\int^t \sqrt{u+1}du$ =? (c) $\int^t \sqrt{u^2+1}udu$ =? (d) $\int^t \sqrt{u+1}udu$

PROOF. In this solution, I will ignore C in intermediate steps. Then, I will add C at the end to the solution. (a) $\int^t udu = \frac{t^2}{2}$ and $\int^t 1du = t$ (consider 1 as u^0). Thus, the answer is $\frac{t^2}{2} + t + C$. (b) Here we need to use the chain rule (theorem D.32(c)). The chain rule is basically the variable substitution rule. First, identify your new variable. $y = u + 1$. Second, express du in terms of the new variable. $dy/du = 1$ and so this means $du = dy$. Third, express the integrand, integral limits and the infinitesimal symbol in terms of the new variable y. $\int^t \sqrt{u+1}du = \int^{t+1} \sqrt{y}dy$. This is now easy to evaluate. The answer is $= \frac{1}{\frac{1}{2}+1}y^{\frac{1}{2}+1}\Big|^{t+1} = \frac{2}{3}(t+1)^{3/2} + C$. (c) The chain rule again. $y = u^2 + 1$. $dy/du = 2u$. So, $dy = 2udu$, or $udu = dy/2$. So, the answer is $\int^{t^2+1} \sqrt{y}\frac{dy}{2} = \frac{1}{3}y^{3/2}\Big|^{t^2+1} = \frac{1}{3}(t^2+1)^{3/2} + C$. (d) Here we use theorem D.32(b), by identifying $F(u) = u$ and $g(u) = \sqrt{u+1}$. $\int^t u\sqrt{u+1}du = u \cdot \frac{2}{3}(u+1)^{3/2}\big|^t - \frac{2}{3}\int^t (u+1)^{3/2}du$ (using the result of (b) for the first term) $= u \cdot \frac{2}{3}(u+1)^{3/2}\big|^t - \frac{2}{3}\frac{2}{5}(u+1)^{5/2}\big|^t$ (the second term has been integrated using the same technique as in (b)) $= \frac{2}{3}t(t+1)^{3/2} - \frac{4}{15}(t+1)^{5/2} + C$. □

D.34. EXAMPLE. [⋆] **Integrals of some useful elementary functions**
Here a, b, α and C are constants. Often in integration tables, C is omitted for convenience but is implied.

- $\int^t du(au+b)^\alpha = \frac{1}{a}\frac{1}{\alpha+1}(at+b)^{\alpha+1} + C$ assuming both LHS and RHS are well-defined for non-integer α
- $\int^t du\cos(au+b) = \frac{1}{a}\sin(at+b) + C$
- $\int^t du\sin(au+b) = -\frac{1}{a}\cos(at+b) + C$
- $\int^t due^{au+b} = \frac{1}{a}e^{at+b} + C$ (also note that e^t and $\exp(t)$ are two notations for the same function)
- $\int^t du\frac{1}{au+b} = \frac{1}{a}\ln(at+b) + C$ assuming $at+b > 0$ and $au+b > 0$