

APPENDIX B

Trigonometry

B.1. DEFINITION. The definition given in Fig. Aii.1 is a general definition of $\sin \theta$ and $\cos \theta$ for *any* real number θ . Then, $\tan \theta \equiv \sin \theta / \cos \theta$ completes the definition of the usual three trigonometric functions.

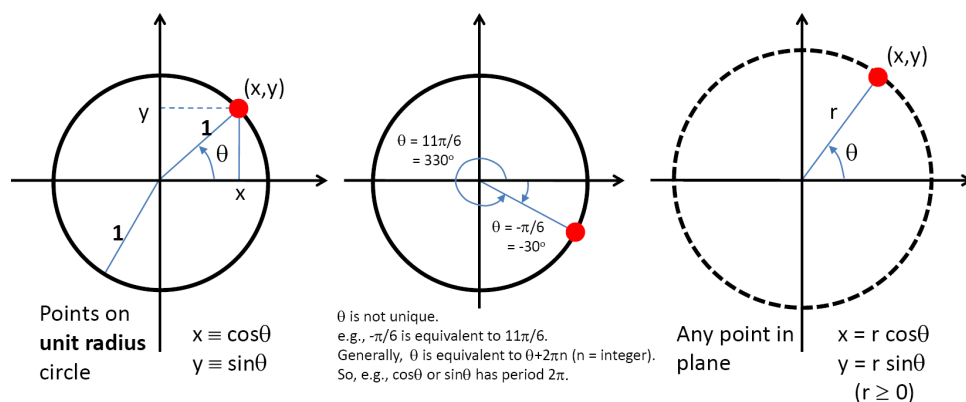


FIGURE AII.1. Definition of $\cos \theta$ and $\sin \theta$, and the representation of points in 2D by r and θ . LEFT: A point (x, y) on a circle of unit radius centered at the origin is defined as $x \equiv \cos \theta$ and $y \equiv \sin \theta$, where the definition of θ is, as shown in the figure, the angle between the positive half of the x axis and the line connecting the point (x, y) and the origin. [Note that this definition agrees with a more elementary definition, based on a triangle, when $0 \leq \theta \leq \pi/2$]. CENTER: θ is defined as positive if the angle is measured counter-clockwise, and negative if measured clockwise. For any given point (x, y) on the unit circle, θ is not unique. It can be positive or negative. It can also make many turns before stopping. Thus, in general, any two angle values different by $2\pi n$ ($n = \text{integer}$) are equivalent. We say " θ is defined modulo 2π ." RIGHT: Given these definitions, note that **any point $\vec{r} = (x, y)$ in 2D can be expressed as**

$$\begin{aligned} x &= r \cos \theta \\ y &= r \sin \theta \end{aligned}$$

where $r = |\vec{r}| = \sqrt{x^2 + y^2}$.

B.2. FACT. *Basic symmetry properties*

$$\sin(-\theta) = -\sin \theta, \quad \cos(-\theta) = \cos \theta, \quad \tan(-\theta) = -\tan \theta$$

$f(\theta + 2\pi n) = f(\theta)$ for $f = \cos, \sin, \tan$ and $n = \text{any integer}$. [Proof left to readers]

B.3. FACT. *Euler formula (optional)*

$$\exp(i\theta) = \cos \theta + i \sin \theta \quad \text{where } i \equiv \sqrt{-1} \quad \text{and } \exp(x) \equiv e^x$$

[Meaning: the 2D plane can be considered as a plane of complex numbers! We then speak of the “complex plane.”]

PROOF. From Taylor expansion: $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} \dots$, $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots$, and $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots$, where $n! = n \cdot (n-1) \cdot \dots \cdot 2 \cdot 1$.

$$\begin{aligned} e^{i\theta} &= 1 + (i\theta) + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \dots \\ &= 1 + i\theta - \frac{\theta^2}{2!} - i\frac{\theta^3}{3!} + \frac{\theta^4}{4!} + \dots \\ &= 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \dots + i \left[\theta - \frac{\theta^3}{3!} + \dots \right] \\ &= \cos \theta + i \sin \theta \end{aligned}$$

□

B.4. FACT. $\frac{d}{dx}e^x = e^x$, $\frac{d}{d\theta} \cos \theta = -\sin \theta$ and $\frac{d}{d\theta} \sin \theta = \cos \theta$ [Note; θ should be in radian]

PROOF. [optional]

$$\frac{d}{dx}e^x = \frac{d}{dx} \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots \right) = 0 + 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots = e^x$$

(using $\frac{d}{dx}x^n = nx^{n-1}$). This means that $\frac{d}{d\theta}e^{i\theta} = i\frac{d}{d(i\theta)}e^{i\theta} = ie^{i\theta} = i\cos \theta + i \cdot i \sin \theta = -\sin \theta + i \cos \theta$.

On the other hand, since $e^{i\theta} = \cos \theta + i \sin \theta$, this last expression must be equal to $\frac{d}{d\theta} \cos \theta + i \frac{d}{d\theta} \sin \theta$. Comparing the real parts and the imaginary parts, we get $\frac{d}{d\theta} \cos \theta = -\sin \theta$ and $\frac{d}{d\theta} \sin \theta = \cos \theta$.

□

B.5. FACT. **Trigonometric identities**

$$\begin{aligned} \cos^2 \theta + \sin^2 \theta &= 1 \\ \sin(\alpha \pm \beta) &= \sin \alpha \cos \beta \pm \cos \alpha \sin \beta \\ \cos(\alpha \pm \beta) &= \cos \alpha \cos \beta \mp \sin \alpha \sin \beta \\ \sin(2\theta) &= 2 \sin \theta \cos \theta \\ \cos(2\theta) &= 2 \cos^2 \theta - 1 = 1 - 2 \sin^2 \theta = \cos^2 \theta - \sin^2 \theta \end{aligned}$$

PROOF. [optional] The first one comes directly from the definition that $(x, y) = (\cos \theta, \sin \theta)$ is on a unit-radius circle centered at the origin, i.e., $x^2 + y^2 = 1$. There are various ways to prove the rest of the identities [and any other trigonometric identities in the literature]. The easiest way to quickly derive these identities, however, is to use Euler formula. Consider, for instance, $e^{i(\alpha+\beta)} = e^{i\alpha}e^{i\beta}$. LHS = $\cos(\alpha + \beta) + i \sin(\alpha + \beta)$. RHS = $(\cos \alpha + i \sin \alpha)(\cos \beta + i \sin \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta + i(\sin \alpha \cos \beta + \cos \alpha \sin \beta)$. By setting LHS = RHS, we get $\cos(\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta$ and $\sin(\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta$, proving two identities above. The rest of the identities above follow from considering two separate cases $\beta \rightarrow -\beta$, and $\alpha = \beta = \theta$ of the two identities that we just proved.

□

B.6. FACT. $\sin \theta \approx \theta$ and $\cos \theta \approx 1 - \frac{1}{2}\theta^2 \approx 1$ for $\theta \approx 0$

[Note: θ should be in radian]

PROOF. Immediate consequence of Taylor series expansion (see proof for Euler formula above). It is important to remember that $\sin \theta \approx \theta \rightarrow 0$ as θ goes to 0, while $\cos \theta \rightarrow 1$ as $\theta \rightarrow 0$.

□