Practice Exam Solutions for mid-term

[1] $\Psi = \sum_n C_n \psi_n$ Show that $\langle \hat{H} \rangle > E_0$

<u>not</u> an eigenstate (Say $n = 0, 1, 2, \ldots$)

$$\langle \hat{H} \rangle = \sum_{m,n} \int dx \, C_n^* \psi_n^* \underbrace{\hat{H} C_m \psi_m}_{\text{eigenstate}} = \sum_{mn} \int dx \, C_n^* \psi_n^* C_m E_m \psi_m$$

$$= \sum_n |C_n|^2 \cdot E_n \geq \sum_n |C_n|^2 \cdot E_0 = E_0 \sum_n |C_n|^2$$

use $\int \psi_n^* \psi_m = \delta_{nm}$

Probability normalization means

$\sum_n |C_n|^2 = 1$ $\therefore \langle \hat{H} \rangle \geq E_0$

If Ψ is not an eigenstate, then $\Psi \neq C_0 \psi_0$ $|C_0| = 1$

Since only when $\Psi = C_0 \psi_0$ $|C_0| = 1$

$$\sum_n |C_n|^2 \cdot E_n = E_0 \sum_n |C_n|^2$$

(assuming $E_1, E_2, \ldots > E_0$)

we have $\langle \hat{H} \rangle > E_0$ for any non-eigenstate

[2] $$-\frac{\hbar^2}{2m} \frac{d^2 \psi}{dx^2} + \frac{m\omega^2}{2} x^2 \psi = E \psi \quad \cdots (*)$$

$$\psi = A e^{-\alpha x^2}$$

$$\psi' = A \cdot (-2\alpha x) e^{-\alpha x^2} = (-2\alpha x) \psi$$

$$\psi'' = \left[(-2\alpha x)^2 - 2\alpha \right] \psi$$

$$= \left[4\alpha^2 x^2 - 2\alpha \right] \psi$$

Plugging this into ($\ast$)

$$\left\{ -\frac{\hbar^2}{2m}\left(4\alpha^2 x^2 - 2\alpha\right) + \frac{m\omega^2}{2}x^2 \right\}\psi = E\psi$$

The x^2 term should vanish:

$$-\frac{\hbar^2}{m}\cdot 4\alpha^2 + m\omega^2 = 0 \Rightarrow \alpha^2 = \frac{m^2\omega^2}{4\hbar^2}$$

The const. term is

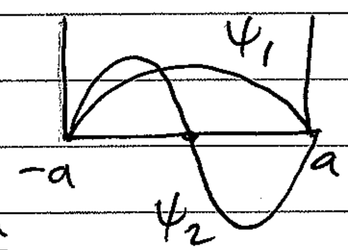
$$\frac{\hbar^2\alpha}{m} = E \qquad \alpha = \frac{m\omega}{2\hbar}$$

$$\therefore E = \frac{1}{2}\hbar\omega \leftarrow \text{as expected.}$$

[3] Particle in a box $\quad \Psi(x,0) = \frac{1}{\sqrt{2}}\left(\phi_1(x) - \phi_2(x)\right)$

In our notation $\quad \phi_n \to \psi_n$

$\psi \to \Psi$



Quantization condition

$$2a = \frac{n\lambda}{2} \qquad \lambda = \frac{4a}{n}$$

$$k = \frac{n\pi}{2a}$$

$$E_1 = \frac{\hbar^2}{2m}\cdot\frac{\pi^2}{(2a)^2}$$

$$E_2 = \frac{\hbar^2}{2m}\cdot\frac{\pi^2}{(2a)^2}\cdot 4$$

Define $\hbar\omega \equiv E_1$

$\Rightarrow E_2 = 4\omega\hbar$

(a) Lowest E that can be measured $= E_1 = \omega\hbar = \dfrac{\hbar^2\pi^2}{2m(2a)^2}$

$$\Psi(x,t) = \frac{1}{\sqrt{2}}\left(\psi_1(x)e^{-i\omega t} - \psi_2(x)e^{-4i\omega t}\right)$$

(b) $\langle \hat{x} \rangle(t) = -\frac{1}{2}\int dx\, \psi_1 \cdot \psi_2 \cdot x \cdot 2\cos(3\omega t)$

$$\psi_1 = \cos\left(\frac{\pi x}{2a}\right)\cdot\sqrt{\frac{1}{a}}$$

You will need to <u>derive</u> this.

$$\psi_2 = \sin\left(\frac{\pi x}{a}\right)\cdot\sqrt{\frac{1}{a}}$$

$$\int dx\, \psi_1 \psi_2\, x = \frac{1}{a}\int_{-a}^{a} dx\, x\, \underbrace{\cos\left(\frac{\pi x}{2a}\right)\sin\left(\frac{\pi x}{a}\right)}_{\frac{1}{2}\left(\sin\frac{3\pi x}{2a} + \sin\frac{\pi x}{2a}\right)}$$

identity <u>will</u> be given

$$\int x \sin x = -x\cos x + \sin x \leftarrow \text{will be given}$$

$$\int dx\, \psi_1\psi_2\, x = \frac{1}{2a}\left[\left(\frac{2a}{3\pi}\right)^2 \cdot 2 \cdot \underbrace{\left(\sin x - x\cos x\right)\Big|_0^{\frac{3\pi}{2}}}_{-1} + \left(\frac{2a}{\pi}\right)^2 \cdot 2 \cdot \underbrace{\left(\sin x - x\cos x\right)\Big|_0^{\frac{\pi}{2}}}_{1}\right]$$

$$= +\frac{4}{a\pi^2}\left(\frac{-1}{9} + 1\right) = +\frac{32}{9\pi^2}a$$

$$\langle \hat{x} \rangle (t) = -\frac{32}{9}\cdot\frac{a}{\pi^2}\cdot\cos(3\omega t)$$

$$\langle \hat{p} \rangle (t) = -i\hbar \int dx\, \Psi^* \frac{\partial \Psi}{\partial x}$$

$$\frac{\partial \Psi}{\partial x} = \frac{1}{\sqrt{2}}\left(\frac{\partial \psi_1}{\partial x} e^{-i\omega t} - \frac{\partial \psi_2}{\partial x} e^{-4i\omega t}\right)$$

$$-\sqrt{\frac{1}{a}}\cdot\frac{\pi}{2a}\sin\left(\frac{\pi x}{2a}\right) \qquad +\sqrt{\frac{1}{a}}\cdot\frac{\pi}{a}\cos\left(\frac{\pi x}{a}\right)$$

$$\langle \hat{p} \rangle (t) = \frac{i\hbar}{2}\int dx \left(\underset{e}{\psi_1^*} e^{i\omega t} - \underset{o}{\psi_2^*} e^{4i\omega t}\right)\left(\sqrt{\frac{1}{a}}\frac{\pi}{2a}\sin\frac{\pi x}{2a}\, e^{-i\omega t} + \sqrt{\frac{1}{a}}\frac{\pi}{a}\cos\frac{\pi x}{a}\, e^{-4i\omega t}\right)$$

$$= \frac{i\hbar}{2}\cdot\frac{1}{a}\cdot\int_{-a}^{a} dx \left\{\cos\left(\frac{\pi x}{2a}\right)\cos\left(\frac{\pi x}{a}\right)\frac{\pi}{a}e^{-3i\omega t} - \sin\left(\frac{\pi x}{a}\right)\sin\left(\frac{\pi x}{2a}\right)\frac{\pi}{2a}e^{3i\omega t}\right\}$$

identity necessary
<u>will</u> be given

$$\int_{-a}^{a} dx \cos\left(\frac{\pi x}{2a}\right)\cos\left(\frac{\pi x}{a}\right) = \frac{1}{2}\int_{-a}^{a} dx \left(\cos\left(\frac{3\pi}{2a}x\right) + \cos\left(\frac{\pi x}{2a}\right)\right)$$

$$= \frac{2a}{3\pi}\sin x\Big|_{0}^{\frac{3\pi}{2}} + \frac{2a}{\pi}\sin x\Big|_{0}^{\frac{\pi}{2}} = \frac{4a}{3\pi}$$

$$\quad -1 \qquad\qquad 1$$

$$\int_{-a}^{a} dx \sin\left(\frac{\pi x}{2a}\right)\sin\left(\frac{\pi x}{a}\right) = \frac{1}{2}\int_{-a}^{a} dx\left(\cos\frac{\pi x}{2a} - \cos\frac{3\pi}{2a}x\right)$$

$$= +\frac{2a}{3\pi} + \frac{2a}{\pi} = +\frac{8a}{3\pi}$$

$$\therefore \langle\hat{p}\rangle(t) = \frac{i\hbar}{2}\cdot\frac{1}{a}\cdot\left(\frac{4}{3}e^{-i3\omega t} - \frac{4}{3}e^{3i\omega t}\right)$$

$$= \frac{-i\hbar 2}{3a}\cdot 2i\sin(3\omega t) = \frac{4\hbar}{3a}\sin(3\omega t)$$

Ehrenfest theorm $\langle\hat{p}\rangle = m\langle\dot{\hat{x}}\rangle$

$$m\langle\dot{\hat{x}}\rangle = +m\frac{32}{3}\cdot\frac{a\,\omega}{\pi^2}\cdot\sin(3\omega t)$$

$$\omega = \frac{\hbar\pi^2}{8ma^2} \Rightarrow \frac{ma\omega}{\pi^2} = \frac{\hbar}{8a}$$

$$\therefore m\langle\dot{\hat{x}}\rangle = \frac{4}{3}\cdot\frac{\hbar}{a}\sin(3\omega t)$$

$$\boxed{\therefore \langle\hat{p}\rangle(t) = m\langle\dot{\hat{x}}\rangle}$$

[4] $\Psi = \frac{1}{\sqrt{2}}(\psi_2 - \psi_0)$ (Don't worry about the Dirac notation of the problem.)

(a) $\Psi(x,t)$?

$E_0 = \frac{1}{2}\hbar\omega \quad E_2 = \frac{5}{2}\hbar\omega$

$$\Psi(x,t) = \frac{1}{\sqrt{2}}\left\{\psi_2(x)\, e^{-i\frac{5}{2}\omega t} - \psi_0(x)\, e^{-i\frac{1}{2}\omega t}\right\}$$

$\langle \hat{x} \rangle (t) =$

$$\hat{a}_\pm = \frac{1}{\sqrt{2m\hbar\omega}}\left(m\omega\hat{x} \pm i\hat{p}\right)$$

$$\Rightarrow \begin{cases} \hat{x} = \sqrt{\frac{\hbar}{2m\omega}}\left(\hat{a}_+ + \hat{a}_-\right) \\ \hat{p} = -i\sqrt{\frac{m\hbar\omega}{2}}\left(\hat{a}_+ - \hat{a}_-\right) \end{cases}$$

$$\hat{x}\,\Psi(x,t) = \frac{1}{\sqrt{2}}\cdot\sqrt{\frac{\hbar}{2m\omega}}\cdot\left[\left(\sqrt{3}\,\psi_3 + \sqrt{2}\,\psi_1\right)e^{-i\frac{5}{2}\omega t} - \psi_1\, e^{-i\frac{1}{2}\omega t}\right]$$

Using $\int \psi_m^* \psi_n = \delta_{mn}$,

$$\int \Psi^* \hat{x}\, \Psi(x,t) = \frac{1}{2}\sqrt{\frac{\hbar}{2m\omega}}\cdot 0 = 0$$

Similarly $\langle \hat{p} \rangle = 0$ at all t

(b) $$\hat{x}^2\,\Psi(x,t) = \frac{\hbar}{2m\omega}\cdot\left(\hat{a}_+ + \hat{a}_-\right)^2 \Psi(x,t)$$

$$= \frac{\hbar}{2m\omega}\cdot\frac{1}{\sqrt{2}}\left[\left(\sqrt{3}\cdot\sqrt{4}\,\psi_4 + \sqrt{2}\cdot\sqrt{2}\cdot\psi_2 + \sqrt{3}\cdot\sqrt{3}\cdot\psi_2 + \sqrt{2}\,\psi_0\right)e^{-i\frac{5}{2}\omega t} - \left(\sqrt{2}\,\psi_2 + \psi_0\right)e^{-i\frac{1}{2}\omega t}\right]$$

Again, using $\int \psi_m^* \psi_n = \delta_{mn}$

$$\int \Psi^* \hat{x}^2 \Psi(x,t) = \frac{\hbar}{2m\omega} \cdot \frac{1}{2} \cdot \left[5 + 1 - \sqrt{2} \cdot 2 \cdot \cos \frac{4}{2} \omega t\right]$$

$$\langle \hat{x}^2 \rangle (t) = \frac{\hbar}{2m\omega} \left[3 - \sqrt{2} \cos(2\omega t)\right]$$

For $\langle \hat{p}^2 \rangle (t)$, simply use

$$\langle \hat{H} \rangle = \text{const} = \frac{3}{2} \hbar\omega = \frac{1}{2} m\omega^2 \langle \hat{x}^2 \rangle + \frac{1}{2m} \langle \hat{p}^2 \rangle$$

$$\frac{1}{2} m \omega^2 \langle \hat{x}^2 \rangle = \frac{3}{4} \hbar\omega - \frac{\hbar\omega}{2\sqrt{2}} \cos(2\omega t)$$

$$\frac{1}{2m} \langle \hat{p}^2 \rangle = \frac{3}{4} \hbar\omega + \frac{\hbar\omega}{2\sqrt{2}} \cos(2\omega t)$$

$$\langle \hat{p}^2 \rangle = m\hbar\omega \left[\frac{3}{2} + \frac{1}{\sqrt{2}} \cos(2\omega t)\right]$$

[5] Free particle

$$\Psi(x,0) = A \sin\left(\frac{2\pi x}{L}\right), \quad -L < x < L$$

(a) $\Psi(x,t) = ?$

$$A = \sqrt{\frac{1}{L}}$$

$\uparrow$ You need to show this

$$\Psi(x,0) = \int dk\, \phi(k) \frac{e^{ikx}}{\sqrt{2\pi}}$$

$$\phi(k) = \frac{1}{\sqrt{2\pi}} \int dx\, e^{-ikx}\, \Psi(x,0)$$

$$= \frac{1}{\sqrt{2\pi}} \cdot A \cdot \int_{-L}^{L} dx\, e^{-ikx} \sin\left(\frac{2\pi x}{L}\right)$$

(Note limits on integral!)

$$= \frac{1}{\sqrt{2\pi}} \sqrt{\frac{1}{L}} \cdot 2 \cdot \int_0^L dx (-i) \sin kx \sin\left(\frac{2\pi x}{L}\right)$$

$$\sin\alpha\sin\beta = \cancel{\sin} \left[\cos(\alpha-\beta) \mp \cos(\alpha+\beta)\right]/2$$

$\nwarrow$ will be given

$$2\int_0^L dx\ \sin(kx)\ \sin\left(\frac{2\pi x}{L}\right) = 2\cdot\frac{1}{2}\int_0^L dx\left\{\cos\left[\left(k-\frac{2\pi}{L}\right)x\right] - \cos\left[\left(k+\frac{2\pi}{L}\right)x\right]\right\}$$

$$= \frac{1}{k-\frac{2\pi}{L}}\sin(kL-2\pi) - \frac{1}{k+\frac{2\pi}{L}}\cdot\sin(kL+2\pi)$$

$$= \frac{1}{k-\frac{2\pi}{L}}(-)\cdot\sin kL - \frac{1}{k+\frac{2\pi}{L}}\cdot\sin(kL)$$

$$= \sin(kL)\cdot\frac{-2k}{k^2+\left(\frac{2\pi}{L}\right)^2}$$

$$\therefore\ \phi(k) = \frac{1}{\sqrt{2\pi}}\cdot\sqrt{\frac{1}{L}}\cdot\sin(kL)\cdot\frac{2ik}{k^2+\left(\frac{2\pi}{L}\right)^2}$$

$$\Psi(x,t) = \int dk\ \phi(k)\cdot\frac{e^{ikx}}{\sqrt{2\pi}}e^{-i\omega_k t},\quad \omega_k = \frac{\hbar k^2}{2m}$$

$$= \frac{i}{\pi}\cdot\sqrt{\frac{1}{L}}\cdot\int_{-\infty}^{\infty}dk\ \frac{k\sin(kL)}{k^2+\left(\frac{2\pi}{L}\right)^2}e^{-i\omega_k t}$$

(b) $\langle\hat{p}\rangle = 0$ since $\Psi(x,0)$ is even w.r.t. $x\to -x$

$$\langle\hat{H}\rangle = -\frac{\hbar^2}{2m}\int\Psi^*\frac{\partial^2}{\partial x^2}\Psi = \text{time-}\underline{\text{indep}}$$

$$= +\frac{\hbar^2}{2m}\cdot\left(\frac{2\pi}{L}\right)^2\cdot\underbrace{\int_{-L}^{L}\Psi^*\Psi}_{=1} = \frac{4\pi^2\hbar^2}{2mL^2}$$

[6] s.h.o.

$$\Psi(x,t) = \sqrt{\frac{2}{5}}\,\Psi_0(x,t) + \sqrt{\frac{3}{5}}\,\Psi_1(x,t)$$

$$E_0 = \frac{1}{2}\hbar\omega_0 \qquad E_1 = \frac{3}{2}\hbar\omega_0$$

(a) No.

(b) $\frac{3}{5}$

(c) $\frac{2}{5}\cdot E_0 + \frac{3}{5}\cdot E_1 = \frac{\hbar\omega_0}{5} + \frac{9\hbar\omega_0}{10} = \frac{11}{10}\hbar\omega_0$

(d) $|\Psi(x,t)|^2 = \frac{1}{5}|\Psi_0|^2 + \frac{3}{5}|\Psi_1|^2 + \sqrt{\frac{6}{25}}\left(\Psi_0^*\Psi_1 + c.c.\right)$

Recall $\Psi_0 = \psi_0 e^{-iE_0 t}$

$\Psi_1 = \psi_1 e^{-iE_1 t}$

$|\Psi_0|^2, |\Psi_1|^2 \rightarrow$ No t-dep.

$\Psi_0^*\Psi_1 + c.c. \rightarrow \propto \cos(E_1 - E_0)t$

$= \cos(\omega_0 t)$

$\therefore\ \omega = \omega_0$ ✓