## Math note



Taking an average is a linear process.

- For a discrete variable $j$ and probability $P_j$
$$\langle \alpha f(j) + \beta g(j) \rangle = \alpha \langle f(j) \rangle + \beta \langle g(j) \rangle$$
- Similarly for a continuous variable $x$ and probability density ftn $\rho(x)$
$$\langle \alpha f(x) + \beta g(x) \rangle = \alpha \langle f(x) \rangle + \beta \langle g(x) \rangle$$

proof.)

$\alpha, \beta$ = constants (i.e. indep of $j$ or $x$)

Only the discrete variable case is discussed here, as the proof for the other case is very similar ("left for homework").

$$\begin{aligned}
\langle \alpha f(j) + \beta g(j) \rangle &= \sum_j \{\alpha f(j) + \beta g(j)\} P_j \\
&= \alpha \sum_j f(j) P_j + \beta \sum_j g(j) P_j \\
&= \alpha \langle f(j) \rangle + \beta \langle g(j) \rangle
\end{aligned}$$

M ①

## Math Trick

① What is $\displaystyle\int_{-\infty}^{\infty} dx\, e^{-x^2}$ ? $\quad\sqrt{\pi}$

② What is $\displaystyle\int_{-\infty}^{\infty} dx\, e^{-\lambda x^2}$ ? $\quad\sqrt{\frac{\pi}{\lambda}}$

③ What is $\displaystyle\int_{-\infty}^{\infty} dx\, x^{2n} e^{-x^2}$ ? $(n \geq 1)$ $\quad\sqrt{\pi}\cdot\frac{1}{2}\cdot\frac{3}{2}\cdot\cdots\cdot\frac{2n-1}{2}$

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① $$\left(\int_{-\infty}^{\infty} dx\, e^{-x^2}\right)^2 = \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy\, e^{-x^2} e^{-y^2} = \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy\, e^{-(x^2+y^2)} = \ast$$

$$\begin{pmatrix} x = r\cos\theta \\ y = r\sin\theta \end{pmatrix} \qquad (r^2 = t)$$

$$\ast = \int_0^{\infty} dr \int_0^{2\pi} d\theta\, r e^{-r^2} = 2\pi \int_0^{\infty} dr\, r e^{-r^2} = \pi \int_0^{\infty} dt\, e^{-t} = \pi$$

QED

② $$\int_{-\infty}^{\infty} dx\, e^{-\lambda x^2} = \frac{1}{\sqrt{\lambda}} \int_{-\infty}^{\infty} dx'\, e^{-x'^2} = \sqrt{\frac{\pi}{\lambda}} \quad \text{by ①}$$

③ $$\int_{-\infty}^{\infty} dx\, x^{2n} e^{-x^2} = \left(-\frac{\partial}{\partial\lambda}\right)^n \cdot \int_{-\infty}^{\infty} dx\, e^{-\lambda x^2}\Bigg|_{\lambda=1} = \left(-\frac{\partial}{\partial\lambda}\right)^n \left(\sqrt{\pi}\cdot\lambda^{-\frac{1}{2}}\right)\Bigg|_{\lambda=1}$$

$$= \sqrt{\pi}\cdot\underbrace{\frac{1}{2}\cdot\frac{3}{2}\cdot\frac{5}{2}\cdots\frac{2n-1}{2}}_{n} \quad \left[= \sqrt{\pi}\cdot\frac{(2n)!}{n!\,2^{2n}}\right]$$

## Math Info

~~Gau~~ Gaussian distribution ftn

$$f(x) = A\, e^{-\frac{x^2}{2\sigma^2}}$$

Normalize : $A \int_{-\infty}^{\infty} dx\, e^{-\frac{x^2}{2\sigma^2}} = 1$

$$A \cdot \sqrt{2}\,\sigma \sqrt{\pi} = 1 \quad \therefore A = \frac{1}{\sigma\sqrt{2\pi}}$$

Mean value : $\langle x \rangle = 0$

$$\langle x^2 \rangle = \int_{-\infty}^{\infty} x^2 f(x)\, dx = A \int_{-\infty}^{\infty} x^2 e^{-\frac{x^2}{2\sigma^2}} dx$$

$$= A \cdot (\sqrt{2}\sigma)^3 \cdot \frac{1}{2} \cdot \sqrt{\pi} = A \cdot \sqrt{2}\,\sigma^3 \sqrt{\pi} = \sigma^2$$

$\therefore$ Standard dev. $= \sqrt{\langle x^2 \rangle - \langle x \rangle^2} = \sigma$

Gaussian Distribution Function

$$\boxed{f(x) = \frac{1}{\sqrt{2\pi}\cdot\sigma}\, e^{-\frac{x^2}{2\sigma^2}}}$$

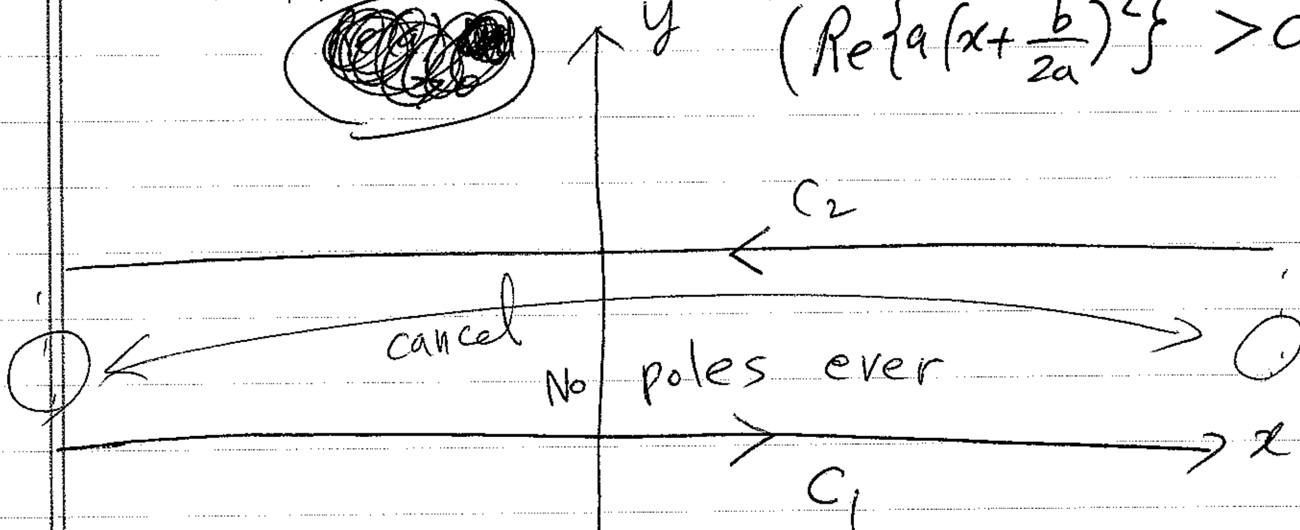
However, warning : do not confuse wave function $\Psi(x)$ and $\rho(x)$.
For $\rho(x)$ to be the normalized gaussian ftn given above
$\Psi(x) = \sqrt{f(x)}$ up to a phase factor.

## Math Trick

$$\int_{-\infty}^{\infty} e^{-(ax^2+bx)}\,dx = \int_{-\infty}^{\infty} dx\, e^{-a\left(x+\frac{b}{2a}\right)^2} e^{\frac{b^2}{4a}}$$

$$= \sqrt{\frac{\pi}{a}} \cdot e^{\frac{b^2}{4a}}$$

Even if $a, b \in \mathbb{Z}$, this holds. $\left( Re\left\{a\left(x+\frac{b}{2a}\right)^2\right\} > 0 \right)$




$$\oint_{C_1} e^{-az^2} = \oint_{C_2} e^{-az^2}$$

from Cauchy theorem of complex analysis.

You are not required to know this.
If you know the theorem, good for you.
If you don't, please take my word for it.
Ref. Arfken etc.

M④

## Math note

$$\psi_n = \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi}{a}x\right), \quad n = \text{integer} > 0$$

$$\int_0^a dx\, \psi_m^* \psi_n = \delta_{mn} = \begin{cases} 1 & m = n \\ 0 & m \neq n \end{cases}$$

proof)

$$I_{mn} = \int_0^a dx\, \psi_m^* \psi_n = \frac{2}{a} \int_0^a dx\, \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi x}{a}\right)$$

$$\underset{\uparrow \atop \frac{\pi x}{a} = y}{=} \frac{a}{\pi} \cdot \frac{2}{a} \int_0^{\pi} dy\, \sin(my) \sin(ny)$$

Now use $\sin\theta = \dfrac{e^{i\theta} - e^{-i\theta}}{2i}$

($\Leftarrow$ Euler relation $e^{i\theta} = \cos\theta + i\sin\theta$)

$$I_{mn} = \frac{2}{\pi} \int_0^{\pi} dy\, \frac{(e^{imy} - e^{-imy})(e^{iny} - e^{-iny})}{-4}$$

$$= -\frac{1}{2\pi} \int_0^{\pi} dy \left\{ e^{i(m+n)y} + e^{-i(m+n)y} - e^{i(m-n)y} - e^{-i(m-n)y} \right\}$$

$$= -\frac{1}{2\pi} \cdot \left( \frac{((-1)^{m+n} - 1) - ((-1)^{m+n} - 1)}{i(m+n)} - \frac{((-1)^{m-n} - 1) - (*)}{i(m-n)} \right)$$

same thing

$= 0$ as long as $m - n \neq 0$

$m + n \neq 0 \Leftarrow$ impossible

If $m = n$, $I_{mn} = 1$

QED.

**Math note.** What in the world does it mean?

① $$\frac{1}{2\pi}\int_{-\infty}^{\infty} dk\, e^{ikx} = \delta(x)$$

② or, $$\frac{1}{2\pi}\int_{-\infty}^{\infty} dx\, e^{ikx} = \delta(k)$$

① is completely equivalent to ②.
Will discuss ②.
② is quite troublesome indeed. When $k \neq 0$, it is clear that $\frac{1}{2\pi}\int_{-\infty}^{\infty} dx\, e^{ikx}$ has many periods of waves that are compensating each other, but why is it necessarily 0? Depending on how one takes limits $\pm\infty$, it can be a finite number!
When $k=0$, clearly $\frac{1}{2\pi}\int_{-\infty}^{\infty} dx\, \underbrace{e^{ikx}}_{=1} \to \infty$ !

A mathematical way to better define the above expression is $$\frac{1}{2\pi}\int_{-\infty}^{\infty} dx\, e^{i(k\pm i\varepsilon)x} \equiv f(k)$$

( $\varepsilon > 0$, $\varepsilon \to 0$, + sign for $x>0$, − sign for $x<0$ )

$$f(k) = \frac{1}{2\pi}\left[\int_{-\infty}^{0} dx\, e^{i(k-i\varepsilon)x} + \int_{0}^{\infty} dx\, e^{i(k+i\varepsilon)x}\right]$$

$$= \frac{1}{2\pi}\left[\frac{1-e^{i(k-i\varepsilon)(-\infty)}}{i(k-i\varepsilon)} + \frac{e^{i(k+i\varepsilon)\infty}-1}{i(k+i\varepsilon)}\right] = \frac{1}{2\pi i}\left(\frac{1}{k-i\varepsilon} - \frac{1}{k+i\varepsilon}\right)$$

See comment in M③. ← From Cauchy's theorem of complex analysis, can show

$$\int_{-\infty}^{\infty} f(k)\, dk = 1$$

With $\varepsilon > 0$, but much smaller than any finite $k$, $\varepsilon \ll |k|$

$f(k) = 0$ for any finite $k$ → $\boxed{\therefore f(k) = \delta(k)}$

Math note M⑥

## §. Dirac delta function $\delta(x)$

**Definition**: $\delta(x) = 0$, $x \neq 0$ and $\displaystyle\int_{-\infty}^{\infty} dx\, \delta(x) = 1$

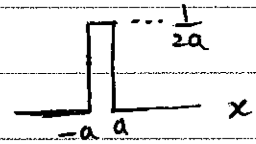
There are various ways to derive $\delta(x)$
[Can show all of them satisfy the above def.]

① $\delta(x) = \theta'(x)$ where $\theta(x) = \begin{cases} 1 & x \geq 0 \\ 0 & x \leq 0 \end{cases}$ → equal sign can be used at either of the two or $\theta(0) \equiv \frac{1}{2}$

$\left[\Rightarrow \delta(x) = \left(\theta(x) + \text{const.}\right)'\right]$

② $\displaystyle \delta(x) = \lim_{\sigma \to 0} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{x^2}{2\sigma^2}\right)$ ← normalized gaussian

③ $\displaystyle \delta(x) = \lim_{a \to 0} \frac{a}{\pi} \cdot \frac{1}{x^2 + a^2}$ ← normalized lorentzian

④ $\displaystyle \delta(x) = \lim_{a \to 0}$ 

← normalized square pulse

Note: $\displaystyle\int_0^{\infty} \frac{\sin x}{x} dx = \frac{\pi}{2}$

⑤ $\displaystyle \delta(x) = \lim_{n \to \infty} \frac{\sin nx}{\pi x}$

⑥ $\displaystyle \delta(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dk\, e^{ikx}$

These two are related since $\displaystyle \frac{1}{2\pi}\int_{-n}^{n} e^{ikx} dk = \frac{\sin nx}{\pi x}$

②–⑤ can be generalized as $\delta(x) =$ limit of any normalized ftn $\int F(x) dx = 1$ with "a width, $W$" of $F(x)$ going to 0.

$\displaystyle \delta(x) = \lim_{W \to 0} F(x)$

①, ⑥ were explained before.

### Properties of $\delta(x)$ proof "left for readers"

① $f(x)\,\delta(x) = f(0)\,\delta(x)$, ② $\delta(-x) = \delta(x)$

③ $\displaystyle \delta(ax+b) = \frac{1}{|a|}\,\delta\left(x + \frac{b}{a}\right)$, ④ $\delta(f(x)) = \left|f'(x)\right|^{-1} \delta(x - x_0)$, $f(x_0) = 0$