L10 – THE HYDROGEN PROBLEM - ADDENDUM FOR DETAILED ALGEBRA

UCSC, PHYS 139A, SPRING 2008
GEY-HONG SAM GWEON

Here are the derviations of the details of the radial equation for the hydrogen problem. The big picture of how things are handled is more important than the exact algebra, which are presented here in full detail for your satisfaction.

Radial Equation for $v(r)$:

How do we go from

$$\frac{d^2u}{d\rho^2} = \left[1 - \frac{\rho_0}{\rho} + \frac{l(l+1)}{\rho^2}\right] u$$

to

$$\rho\frac{d^2v}{d\rho^2} + 2(l+1-\rho)\frac{dv}{d\rho} + [\rho_0 - 2(l+1)]v = 0$$

using $u(\rho) = \exp(-\rho)\rho^{l+1}v(\rho)$?

Proof. First

$$u' = [-\rho + (l+1)]\, e^{-\rho}\rho^l v + e^{-\rho}\rho^{l+1}v'$$

So

$$\begin{aligned} u'' &= -e^{-\rho}\rho^l v + [\rho - (l+1)]\, e^{-\rho}\rho^l v \\ &\quad + l\,[-\rho + (l+1)]\, e^{-\rho}\rho^{l-1}v + [-\rho + (l+1)]\, e^{-\rho}\rho^l v' \\ &\quad - e^{-\rho}\rho^{l+1}v' + (l+1)e^{-\rho}\rho^l v' + e^{-\rho}\rho^{l+1}v'' \end{aligned}$$

By collecting terms of factoring out $e^{-\rho}\rho^{l-1}v$, and $e^{-\rho}\rho^l v'$, we see

$$\begin{aligned} u'' &= e^{-\rho}\rho^{l-1}v\left[-\rho + \rho^2 - (l+1)\rho - l\rho + l(l+1)\right] \\ &\quad + e^{-\rho}\rho^l v'\left[-\rho + (l+1) - \rho + (l+1)\right] \\ &\quad + e^{-\rho}\rho^{l+1}v'' \\ &= e^{-\rho}\rho^{l-1}v\left[\rho^2 - 2(l+1)\rho + l(l+1)\right] \\ &\quad + e^{-\rho}\rho^l v'\left[2(l+1) - 2\rho\right] \\ &\quad + e^{-\rho}\rho^{l+1}v'' \end{aligned}$$

Setting this equal to

$$\left[1 - \frac{\rho_0}{\rho} + \frac{l(l+1)}{\rho^2}\right] u = e^{-\rho}\rho^{l-1}v[\rho^2 - \rho_0\rho + l(l+1)]$$

we see that the differential equation becomes

$$e^{-\rho}\rho^{l-1}v\left[\rho_0\rho - 2(l+1)\rho\right] + e^{-\rho}\rho^l v'\left[2(l+1) - 2\rho\right] + e^{-\rho}\rho^{l+1}v'' = 0$$

which means

$$\rho v'' + 2\left[(l+1) - \rho\right] v' + \left[\rho_0 - 2(l+1)\right] v = 0 \qquad (0.1)$$

which is what we wanted to get! Bravo! □

How do we get that recursion relation?

Plug in $v(\rho)=\sum_{j=0}^{\infty} c_j\rho^j$. $v'=\sum_{j=1}^{\infty} c_j j\rho^{j-1}$ $v''=\sum_{j=2}^{\infty} c_j(j)(j-1)\rho^{j-2}$. Thus

$$\begin{aligned}
\rho v'' &= \sum_{j=2} c_j j(j-1)\rho^{j-1} \\
(j\to j+1) &= \sum_{j=1}^{\infty} c_{j+1}(j+1)j\rho^j \\
(j=0\, term\, is\, 0\, anyway) &= \sum_{j=0}^{\infty} c_{j+1}(j+1)j\rho^j
\end{aligned}$$

$$\begin{aligned}
2\left[(l+1)-\rho\right]v' &= 2(l+1)\sum_{j=1}^{\infty} c_j j\rho^{j-1} - 2\sum_{j=1}^{\infty} c_j j\rho^j \\
(j\to j+1\, for\, the\, first\, term) &= 2(l+1)\sum_{j=0}^{\infty} c_{j+1}(j+1)\rho^j - 2\sum_{j=1}^{\infty} c_j j\rho^j \\
(j=0\, for\, the\, 2nd\, term\, is\, 0\, anyway) &= \sum_{j=0}^{\infty}\left[2(l+1)(j+1)c_{j+1}-2jc_j\right]\rho^j
\end{aligned}$$

$$\left[\rho_0-2(l+1)\right]v=\sum_{j=0}^{\infty} c_j\left[\rho_0-2(l+1)\right]\rho^j$$

Eq. 0.1 means that summing up all three of these series expansions should be 0 for all ρ values. The sum of the three results is

$$\sum_{j=0}^{\infty}\rho^j\left\{c_{j+1}(j+1)j+2(l+1)(j+1)c_{j+1}-2jc_j+c_j\left[\rho_0-2(l+1)\right]\right\}$$

which is equal to

$$\sum_{j=0}^{\infty}\rho^j\left\{c_{j+1}(j+1)(j+2l+2)+c_j(\rho_0-2(j+l+1))\right\}$$

In order for this to zero for all ρ, each term should be zero. Thus $c_{j+1}(j+1)(j+2l+2)+c_j(\rho_0-2(j+l+1))=0$ which means

$$c_{j+1}=\left\{\frac{2(j+l+1)-\rho_0}{(j+1)(j+2l+2)}\right\}c_j \tag{0.2}$$