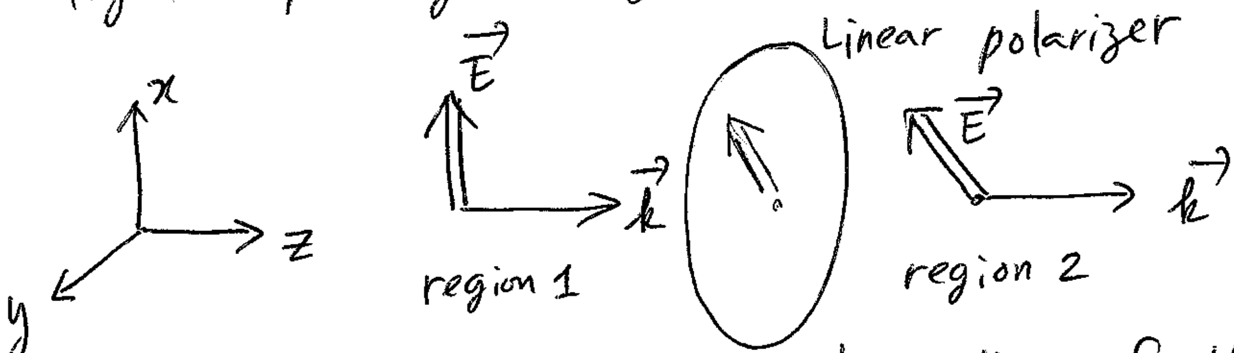
## § Introduction – Superposition

Superposition is a familiar concept from Optics. Two waves can co-exist and form a superposed state.

Here we show an example from a Classical view and then discuss the Quantum limit.

Experiment to consider is a linearly polarized light passing through a linear polarizer.




As defined above, the polarization of the light before entering the polarizer is along the x axis by definition. We write

$$\vec{E} = E\,\vec{\varepsilon}_p\, e^{i(\vec{k}\cdot\vec{x}-\omega t)} \quad \text{... region 1}$$

$$\underbrace{\vec{\varepsilon}_p}_{\text{polarization vector of light (unit vector)}} = \underbrace{\vec{e}_1}_{\text{unit vector along the x direction.}}$$

$$\vec{E}' = E'\,\vec{e}_M\, e^{i(\vec{k}\cdot\vec{x}-\omega t)} \quad \text{... region 2}$$

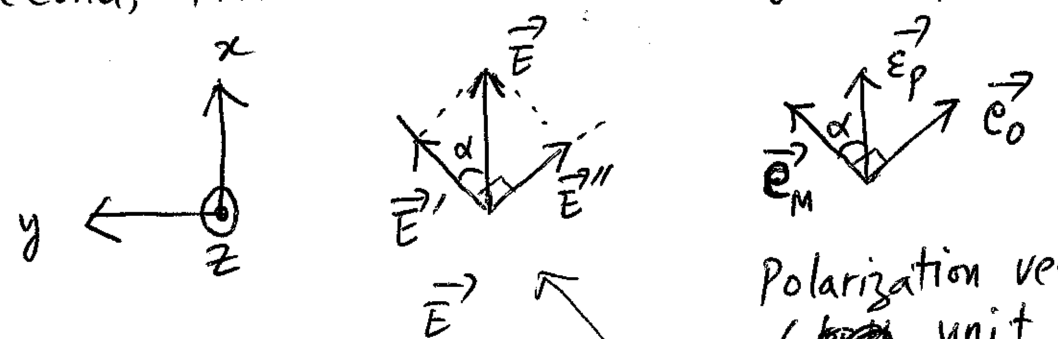
$$\vec{e}_M \equiv \text{polarization direction of the polarizer}$$

$$\vec{e}_M = \vec{e}_1 \cos\alpha + \vec{e}_2 \sin\alpha$$

Question I What is the relative intensity that goes through the polarizer? I.e., for a given incident intensity of 1, how much is the intensity in region 2?

Answer I $\cos^2\alpha = (\vec{e}_M \cdot \vec{\varepsilon}_p)^2$

why? First of all, the intensity is given by the Poynting vector $\vec{S} = \vec{E} \times \vec{H} = \frac{1}{\mu_0}\frac{n}{c}|E|^2 \vec{e}_3$
so intensity $\propto |\vec{E}|^2$. ($n = 1$ in vac)
Second, from the definition of the problem




Polarization vectors
(all unit size by def.)

the initial wave can be thought of as a superposition of two waves with $\vec{E}'$ and $\vec{E}''$ with polarization vectors $\vec{e}_M$ and $\vec{e}_0$ respectively.
Finally, only $\vec{E}'$ with the correct polarization passes through, and from the diagram it is clear that $|\vec{E}'| = |\vec{E}|\cos\alpha$
So, the relative intensity that goes through is

$$\frac{|\vec{E}'|^2}{|\vec{E}|^2} = \cos^2\alpha$$

Noting that $\cos\alpha = \vec{\varepsilon}_p \cdot \vec{e}_M$ we can equate this to $(\vec{e}_M \cdot \vec{\varepsilon}_p)^2$

So far it is just good old Optics or E&M. Quantum Mechanics kicks in as we imagine reducing the intensity of the incoming light lower and lower.

What people have discovered is that as the intensity is lowered the <u>energy</u> of light is not continuous any more but <u>quantized</u> as

$$\begin{aligned} &\text{integer} \times \hbar\omega \\ &= \text{integer} \times h\nu \end{aligned}$$

$\hbar = 1.05457 \times 10^{-34}$ J s (Planck constant)

$\omega = \frac{2\pi}{T}$, $\nu = \frac{1}{T}$ ($T$ = period)

Let us consider an incoming light with just one quantum of energy $\hbar\omega$ and ask the

<u>Question II</u>: What happens to <u>this light</u> (one quantum) when it goes through the polarizer?

[ Note that there is some tricky business of thinking about the energy and the intensity. When we say that the energy of the light is quantized, we mean the energy density of light integrated over all space. Here we consider light that is reasonably localized so it makes sense to talk about light being in region 1 or 2, while locally light is pretty much plane wave like --- i.e, we are considering a wave packet, which we will define rigourously later. ]

Before giving the answer, it is important to note one thing: as the energy of the light increases from one quantum to N quantum
($N \gg 1$)
all that is happening is $|\vec{E}|^2 \propto N$ increases and <u>nothing else</u>. You can visualize this as the amplitude of the wave increasing as more and more quanta of energy is harnessed by the wave. When I say nothing else happens, I mean the other characteristics of the wave --- wave vector, frequency --- remain intact.

In the <u>particle</u> language, the quantum of energy <u>defines</u> the "<u>photon</u>", the light particle. The above statement about nothing else happening means that photons do not interact with each other, a pretty good approximation for light with low energy (visible light, e.g.).

This important observation leads to the following.

<u>Expt 1</u> Send N photons at once thru the polarizer.
($N \to \infty$ means classical limit)

<u>Expt 2</u> Send 1 photon N times and take an average.

Expt 1 and 2 are completely equivalent!
as far as computing the relative intensity that goes thru.

To answer Question (II), let us say that there is a <u>probability</u> $P$ of one photon to go through.
By the equivalence of Expt 1 and 2

$$\cos^2\alpha = (\vec{e}_M \cdot \vec{\varepsilon}_p)^2 = P \cdot N \cdot \frac{1}{N}$$

Answer (I) Classical Optics

$P$: Quantum Optics! $N$: repeat $\frac{1}{N}$: average

$$\boxed{P = (\vec{e}_M \cdot \vec{\varepsilon}_p)^2 = \cos^2\alpha}$$

We made a tremendous jump in interpreting the same formula.

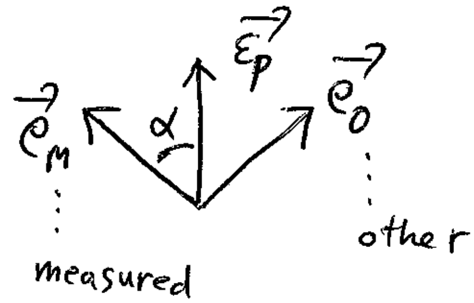
<u>Note 1</u> We <u>have to</u> introduce the concept of the probability of the photon ~~that~~ to go through, because its energy cannot be reduced ~~to~~ below one quantum (except zero).

<u>Note 2</u> By introducing the probability $P$, we have a very uncomfortable situation – <u>indeterminacy</u>!
We cannot tell what will happen to the photon for sure in general (unless $P = 0$ or $1$).

(See Note 4 below!)

Note 3 $\vec{e}_M$ and $\vec{e}_0$ $\vec{\varepsilon}_p$ are examples of **wave functions.**

Unit length polarization vectors




$$\vec{\varepsilon}_p = \vec{e}_M \cos\alpha + \vec{e}_0 \sin\alpha$$

When "measured" by the polarizer the wave function "collapses" to $\vec{e}_M$ with probability $P = (\vec{e}_M \cdot \vec{\varepsilon}_p)^2$

$\vec{e}_M \cdot \vec{\varepsilon}_p$ = Component of wave function $\vec{\varepsilon}_p$ along $\vec{e}_M$

= Probability Amplitude

[ Note that $\vec{\varepsilon}_p$ is a 2-dim vector characterized by two numbers $(\cos\alpha, \sin\alpha)$. This is because there are only two possible polarizations. Other types of wave functions $\psi(x)$ are characterized by infinite numbers since there are infinite possible values of $x$. ]

If circular polarizers are considered then $P$ would be $|\vec{e}_M^{\,*} \cdot \vec{\varepsilon}_p|^2$. → Complex pol. vectors for circ. pol.

In any case it is the general feature of QM that

$$\text{Probability} = |\text{Probability Amplitude}|^2$$

$$= |\text{Wave function Value/Component}|^2$$

**Note 4** The weird indeterminacy is the feature of any experiment involving a measurement of a quantum object.
Note that measurement always involves a large equipment that is not quantum object.
So the

weirdness
indeterminacy
wave function collapse
uncertainty...

happen because of the interaction of a Quantum ~~QM~~ system & a Classical system.

QM itself (is not weird and) is actually very deterministic (Knowing wavefunction at $t=0$ tells everything about what will happen at a later time for a given Hamiltonian).

**Note 5** In QM, not only polarization but all other dynamical variables (momentum, energy, position, ...) have the superposition properties! For any particle.

(electron
nucleon
photon
⋮)

Natural for Waves.

QM = Wave Mechanics.