Homework 8

Phys 139A, Spring 2008, UCSC

Due June 6, 1 pm

1. [Radial wave functions; 10 points] The famous 21 cm microwave radiation prevalent in the Universe is due to a minute splitting of the 1s level of the Hydrogen, called "hyperfine splitting." This energy splitting is due to the interaction of the electron spin and the proton spin. In general, for a stationary state wavefunction for the electron ψ, the hyperfine splitting results from the "contact" between the proton spin and the electron spin at the nucleus, and as such is proportional to $|\psi(0)|^2$. [It is the 2nd term of Eq. 6.88 of Griffiths, which is the only non-vanishing term in the ground state.] In our solution to the radial equation of the Hydrogen problem, we wrote
$$u_{nl}(\rho) = \exp(-\rho)\rho^{l+1}v_{nl}(\rho)$$
where $v(\rho) = \sum_{j=0}^{j_{max}} c_j \rho^j$, where $\rho = \kappa_n r$ (Eq. 1.3 of L10, or Eq. 4.60 of Griffiths). Prove that a stationary state wave function $\psi_{nlm}(x, y, z) = R_{nl}(r)Y_{lm}(\theta, \phi)$ where $R = u/r$ can give a non-zero contribution to the hyperfine splitting, only if ψ_{nlm} is an s ($l = 0$) wave function. (Here I am *not* talking about the $1s$ orbital alone, but speaking generally for all s states, like $2s, 3s, \ldots$. This general consideration is very important for some experimental techniques such as NMR.) In doing this problem, the only thing that you need to know/assume about Y_{lm} is that it is finite for any θ, ϕ, i.e. it is not a divergent function. You only need to show that if $l \neq 0$ then the hyperfine splitting vanishes.

2. [Basic commutation relations in three dimensions; 15 points] Problem 4.1

3. [Wave functions for Hydrogen; 15 points] Problem 4.13. Note that for this problem (like many problems in three dimensions) the skill to look up tables is essential. Namely, to know $\psi_{nlm}(x, y, z) = R_{nl}(r)Y_{lm}(\theta, \phi)$, you need to look up R_{nl} and Y_{lm} (Y_l^m in the textbook notation) in table 4.7 and table 4.3, respectively. Then, you write the function down, and do appropriate three-dimensional integrals. Final results to part (a) can be inferred from notes at the end of L10. However, you need to show your work leading to those results. Recall $\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} dz[...] = \int_0^{\infty} dr r^2 \int_0^{\pi} d\theta \sin\theta \int_0^{2\pi} d\phi[...]$.

4. [Three dimensional harmonic oscillator; 15 points] Problem 4.38

5. [Angular momentum; 15 points] Let us consider a general angular momentum operator $\hat{\vec{J}}$, whose components are $\hat{J}_x$, $\hat{J}_y$, $\hat{J}_z$. As usual, consider the state $|jm\rangle$ which is a simultaneous eigenstate of $\hat{J}^2$ and $\hat{J}_z$ such that $\hat{J}^2|jm\rangle = j(j+1)\hbar^2|jm\rangle$ and $\hat{J}_z|jm\rangle = m\hbar|jm\rangle$. Prove that, for $|jm\rangle$,
$$\langle \hat{J}_x \rangle = \langle \hat{J}_y \rangle = 0$$
and
$$\langle \hat{J}_x^2 \rangle = \langle \hat{J}_y^2 \rangle = \left[j(j+1) - m^2\right] \hbar^2/2$$
[Hint: Express $\hat{J}_{x,y}$ in terms of $\hat{J}_\pm$. You may not need even that to prove the relation for $\langle \hat{J}_{x,y}^2 \rangle$, if you carefully argue about the arbitrariness of defining xy axes.]