Homework 7

Phys 139A, Spring 2008, UCSC

Due May 30, 5 pm

Spin – Quantum Mechanics in a Nutshell

[100 points total]

A spin 1/2 problem illustrates most of the basic principles of Quantum Mechanics, with the least complications. The Hilbert space for this problem is a two dimensional inner product space, where a general vector $|\alpha\rangle = \begin{bmatrix} u \\ v \end{bmatrix}$ where u and v are complex numbers. The following three "Pauli matrices" play a fundamental role in the spin physics, as they represent x, y, z components of the spin 1/2 angular momentum [up to the constant multiplication factor of $\hbar/2$].

$$\sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \sigma_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

All problems below are interconnected with each other, while it is possible to do some later problems without knowing answers to earlier problems. The mathematical complexity of these problems is not high at all. All necessary mathematics is summarized in the last page of this document. In particular, if you need a quick refreshment course in Linear Algebra, I recommend reading that page first.

Note that I do not use ˆ for Pauli matrices, but they *are* obviously QM operators, being matrices. Since they cannot be confused with numbers, I will take hats from them to be efficient. I will still keep ˆ on spin though.

1. Is σ_x a Hermitian operator? How about σ_y and σ_z? Now, fill in the blanks of the following expressions. Each blank can be filled by one of $0, 1, \sigma_x, \sigma_y, \sigma_z$ and nothing else. You need to show your work, not just answers, to get the credit.

$$[\sigma_x, \sigma_y] = 2i(\ \ ),\ [\sigma_y, \sigma_z] = 2i(\ \ ),\ [\sigma_z, \sigma_x] = 2i(\ \ )$$

$$\sigma_x^2 = (\ \ ),\ \sigma_y^2 = (\ \ ),\ \sigma_z^2 = (\ \ )$$

$$\sigma^2 = \sigma_x^2 + \sigma_y^2 + \sigma_z^2 = 3(\ \ )$$

$$[\sigma^2, \sigma_x] = (\ \ ),\ [\sigma^2, \sigma_y] = (\ \ ),\ [\sigma^2, \sigma_z] = (\ \ )$$

   Is σ^2 a Hermitian operator?

2. Find eigenvalues and eigenstates for each Pauli matrix. Verify that eigenvalues are 1 or -1 for each Pauli matrix. Be sure to normalize eigenstates. Are the eigenstates of a given Pauli matrix orthogonal to each other? Explain why your answer was to be expected based on the Hermitian nature (or not) of Pauli matrices. Let $B_x = \{|x \uparrow\rangle, |x \downarrow\rangle\}$, $B_y = \{|y \uparrow\rangle, |y \downarrow\rangle\}$, $B_z = \{|z \uparrow\rangle, |z \downarrow\rangle\}$, where x, y, z means $\sigma_x, \sigma_y, \sigma_z$ respectively, and $\uparrow$ means eigenvalue 1 and $\downarrow$ means eigenvalue -1. Does each/any of the sets B_x, B_y, B_z form a natural basis (Definition 4.9 of L8) for the Hilbert space defined above?

[Partial answer as hint] $\sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$: det $\begin{bmatrix} -s & 1 \\ 1 & -s \end{bmatrix} = 0$ gives $s^2 - 1 = 0$. $s = \pm 1$. For $s = 1$, the eigenstate is given by $\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = 0$, which means $\alpha = \beta$. A normalized solution can be taken as, then,

$$|x\uparrow\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Similarly,

$$|x\downarrow\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

3. The quantiy $|\langle a\alpha|b\beta\rangle|^2$ is a constant for any possible combinations of $a \neq b$, $a, b = x, y, z$, $\alpha = \uparrow, \downarrow$, $\beta = \uparrow, \downarrow$. There are twelve distinct combinations possible. Show your work to calculate $\langle a\alpha|b\beta\rangle$ for at least six combinations of $a \neq b, a, b, \uparrow, \downarrow$, and find what that constant is.
   Some partial answers: $\langle z\uparrow|x\uparrow\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{2}}$, $\langle z\uparrow|x\downarrow\rangle = \frac{1}{\sqrt{2}}$, $\langle z\downarrow|x\downarrow\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = -\frac{1}{\sqrt{2}}$

4. Suppose you prepared a particle in the $|z\uparrow\rangle$ state. Then, you measure σ_y. What is the probability that a measurement will yield the value 1 (σ_y is "up") or -1 (σ_y is "down")? Verify that the sum of the two probabilities is 1. Consider the measurement event in which $\sigma_y = 1$ is measured. Write down the wave function after this measurement. Find the 2×2 matrix corresponding to the $\hat{h}$ operator for this measurement, $\hat{h} = |\Psi_{new}\rangle\langle\Psi_{new}|$, as defined in Theorem 5.4 of L8. Let us call this operator $\hat{h}_{y\uparrow}$. Verify explicitly that the 2×2 matrix for $\hat{h}_{y\uparrow}$ is Hermitian. Find also, the 2×2 matrix for $\hat{h}_{y\downarrow}$. What is $\hat{h}_{y\uparrow} + \hat{h}_{y\downarrow}$? How is this related to Theorem 4.15 of L8?

5. For a magnetic dipole moment $\vec{\mu}$ in a $\vec{B}$ field, the Hamiltonian is given by $\hat{H} = -\vec{\mu}\cdot\vec{B}$. Let us consider a constant field $\vec{B} = B\vec{k}$ where $\vec{k}$ is the unit vector along the z direction. For the spin system $\vec{\mu} \propto -\vec{\sigma}$, and so we can write the Hamiltonian as

$$\hat{H} = \hbar\omega_L\sigma_z/2$$

   where ω_L is a frequency scale $\propto B$. Find eigenstates (i.e. stationary states in this case) of $\hat{H}$ and corresponding energy eigenvalues E_0 and E_1. For an arbitrary initial state $|\Psi(t=0)\rangle = \begin{bmatrix} u \\ v \end{bmatrix}$, find $|\Psi(t)\rangle$ using the standard procedure: i.e. by first expressing $|\Psi(t=0)\rangle$ as a linear combination of stationary states and then applying $\exp(-iE_nt/\hbar)$ to each stationary state. For simplicity, let us assume u, v are *real* numbers. It turns out that using this assumption does not restrict physics in any way [can you tell why by looking at your solution for $|\Psi(t)\rangle$?].
   This problem needs very small amount of work, since $\hat{H} \propto \sigma_z$.

6. From the solution of prob. 5, calculate $\langle\sigma_x\rangle$, $\langle\sigma_y\rangle$, and $\langle\sigma_z\rangle$ *as a function of time*, and discuss what the $\vec{B}$ field is doing on the spin. Since $u^2 + v^2 = 1$, by normalization and the real nature assumed for u, v, it will help to put $u = \cos(\theta_0/2)$ and $v = \sin(\theta_0/2)$, *after* you are done obtaining simple forms for the expectation values in terms of u, v, ω_L, t. What is the relationship between $|\Psi(t = 2\pi/\omega_L)\rangle$ and $|\Psi(t=0)\rangle$? [How do you make peace with that relationship?]

7. Calculate $[\hat{H}, \sigma_x]$, $[\hat{H}, \sigma_x]$, and $[\hat{H}, \sigma_z]$, using the results of prob. 1. In light of Definition 8.3 of L8, which of $\sigma_x, \sigma_y, \sigma_z$ is/are conserved quantity/quanties? Briefly discuss your answers to prob. 6 from this perspective. [Note that $\partial\sigma_x/\partial t = \partial\sigma_y/\partial t = \partial\sigma_z/\partial t = 0$ *by definition*, since in the angular momentum physics, angle, angular momentum, and time are totally independent variables *in the language of multi-dimensional calculus, not necessarily in the sense of quantum mechanics.* In quantum mechanics, angle and angular momentum are not independent dynamical variables.]
`This problem needs very small amount of work, since` $\hat{H} \propto \sigma_z$.

8. Show *qualitatively* that your answers to prob. 6 is consistent with the energy time uncertainty relation (Theorem 7.5 of L8), taking the case $|u| \sim |v|$ as an example. [Hint: The word "qualitatively" means that you can just show that $\Delta E \Delta t \gtrsim \hbar$, without worrying too much about numerical factors. Namely, you can do similarly as in Example 3.5 of the textbook, *without* using the precise definitions of ΔE and Δt.]
`Hints: What is the rough width in energy? What is the typical time scale in which` $\langle\sigma_x\rangle$, $\langle\sigma_y\rangle$, `and` $\langle\sigma_z\rangle$ `changes? Example 3.5 will help you a lot, if you are unsure.`

9. For this part, use $\theta_0 = \pi/2$ for simplicity, so that $u = v = 1/\sqrt{2}$. Verify the generalized uncertainty relation $\Delta\hat{A}\Delta\hat{B} \geq \left|\frac{1}{2i}\langle[\hat{A}, \hat{B}]\rangle\right|$ at *any* t, if you take $\hat{A}, \hat{B}$ as any two observables picked from $\sigma_x, \sigma_y, \sigma_z$. Note that there are times at which $\Delta\hat{A}\Delta\hat{B} = 0$. Discuss the nature of the state at those times using $\Delta\sigma_x, \Delta\sigma_z, \Delta\sigma_z$. Note that $\Delta\hat{A}\Delta\hat{B} = 0$ does not necessarily mean that the two observables $\hat{A}, \hat{B}$ are precisely determined at the same time. [Hint: probs. 1,3]

10. Suppose you prepared two particles in the same state $|z\uparrow\rangle$. You apply measurement sequence 1 to the first particle, and measurement sequence 2 to the other particle.

$$\text{sequence 1} : \sigma_y\sigma_z\sigma_y\sigma_z$$
$$\text{sequence 2} : \sigma_x\sigma^2\hat{H}\sigma_z$$

Here, $\hat{H}$ is still that of prob. 5. Make a table indicating a set of possible outcomes and whether or not there would be a wave function collapse at each measurement step of a given sequence. Wave function collapse means that the wave function goes through more than just a possible overall phase change as a consequence of the measurment event. A concise condition for a wave function collapse is that $|\langle\Psi_{new}|\Psi_{old}\rangle|$ is strictly less than 1. One example is given below as to how to fill in the table. In the last column write the total number of distinct outcome sequences. For instance, in measurement sequence 1, an outcome sequence ↑↑↓↑ counts as 1, and you need to count all possible outcome sequences like that. [Hint: Definition 7.2 of L8, probs. 1,3,7]
`Hints: If their commutator is zero, then two operators are compatible. If not, not.`

measurement	σ_y	σ_z	σ_y	σ_z	Number of distinct outcome sequences
possible outcomes	↑, ↓				
wave function collapse	Y				N/A

Linear Algebra in a Nutshell

Linear Algebra in a 2 dimensional Hilbert space can give you a lot of mileage in various physics problems. Here is a list of essential things to know.

Vector

$|\alpha\rangle = \begin{bmatrix} u \\ v \end{bmatrix} \qquad \langle\alpha| = \begin{bmatrix} u^* & v^* \end{bmatrix}$

$\langle\alpha|\alpha\rangle = u^*u + v^*v$ The length/norm of $|\alpha\rangle = \sqrt{\langle\alpha|\alpha\rangle}$.

$\langle\alpha|\beta\rangle = v^*w + v^*x$ if $|\beta\rangle = \begin{bmatrix} w \\ x \end{bmatrix}$

$\langle\alpha|\alpha\rangle\langle\beta|\beta\rangle \geq |\langle\alpha|\beta\rangle|^2$

Matrix

$\hat{O} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad \hat{O}^\dagger = \begin{bmatrix} a^* & c^* \\ b^* & d^* \end{bmatrix}$

$\hat{O}^\dagger = \hat{O}$? Then, $\hat{O}$ is Hermitian.
$\hat{O}^\dagger = \hat{O}^{-1}$? Then, $\hat{O}$ is unitary.
$\hat{O}$ is unitary if and only if $\langle\alpha|\alpha\rangle = \langle\beta|\beta\rangle$ for all $|\alpha\rangle$ and $|\beta\rangle = \hat{O}|\alpha\rangle$.
Hermitian and unitary (in general, normal, $[\hat{N}, \hat{N}^\dagger] = 0$) matrices are always diagonalizable.
Hermitian matrix has orthogonal eigenvectors.
Eigenvalues of a Hermitian matrix are real numbers.
Eigenvalues of a unitary matirx have magnitude 1, i.e. $\exp(i\theta)$ for some real number θ.

Eigenvalue Problem

For a given operator $\hat{O}$, the eigenvalue problem is defined as

$$\hat{O}|\lambda\rangle = \lambda|\lambda\rangle$$

where $|\lambda\rangle \neq 0$. Putting $|\lambda\rangle = \begin{bmatrix} y \\ z \end{bmatrix}$, the matrix form of this equation is

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} y \\ z \end{bmatrix} = \lambda \begin{bmatrix} y \\ z \end{bmatrix}$$

which can be rewritten as

$$\begin{bmatrix} a-\lambda & b \\ c & d-\lambda \end{bmatrix} \begin{bmatrix} y \\ z \end{bmatrix} = 0 \tag{1}$$

The condition for this equation to be solvable for non-zero $|\lambda\rangle$ is

$$\det \begin{bmatrix} a-\lambda & b \\ c & d-\lambda \end{bmatrix} = (a-\lambda)(d-\lambda) - bc = 0 \tag{2}$$

For each λ value obtained by solving this eigenvalue equation, or "secular equation," the corresponding eigenvector $|\lambda\rangle$ can be obtained by solving Eq. 1 for y, z. The two linear equations $(a-\lambda)y + bz = 0$ and $cy + (d-\lambda)z = 0$ are equivalent (namely, if they are not equivalent, that tells you that you made mistakes somewhere), by the very fact that the determinant is zero (Eq. 2), and so you have only one equation to solve for two complex numbers y, z. Therefore, all you get is the relationship between y and z. Even after normalizing according to $\langle\lambda|\lambda\rangle = 1$, i.e. $yy^* + zz^* = 1$, the overall phase of $|\lambda\rangle$ remains arbitrary. Then, it is your call how to fix the phase, e.g. one is free to choose the first non-zero component of $|\lambda\rangle$ to be a positive number.