Homework 6

Phys 139A, Spring 2008, UCSC

Due May 23, 1 pm

1. [10 points; Some properties of commutator] Problem 3.13
2. [20 points; Conservations and Ehrenfest Theorem] Problem 3.17
3. [15 points; Virial theorem] Problem 3.31
4. [15 points; Scattering Matrix] Consider problem 2.52. Keep in mind that all numbers represented by various symbols (A, B, S_{ij},etc.) are complex.

   (a) By considering particle conservation, show that the S matrix is a unitary matrix. You would need to show that the norms of two vectors $\begin{pmatrix} A \\ G \end{pmatrix}$ and $\begin{pmatrix} B \\ F \end{pmatrix}$ are always the same.

   (b) Consider two column vectors $\begin{pmatrix} S_{11} \\ S_{21} \end{pmatrix}$ and $\begin{pmatrix} S_{12} \\ S_{22} \end{pmatrix}$. Show that the unitarity of the S matrix is equivalent to the condition that these two vectors are orthonormal to each other.

   (c) Consider the scattering state solutions for the delta function well problem and the finite square well problem [text or lecture note]. In that case, we considered $G = 0$ case, and I put A=1. By obtaining solutions for this problem, we have, according to Equation [2.175], obtained solutions for S_{11} and S_{21}. Write down what these are for the two problems, simply making use of the solutions given in the text or the lecture note. If we now put $A = 0$ and $G = 1$, and do the same problems all over gain, we would get the other column vector $\begin{pmatrix} S_{12} \\ S_{22} \end{pmatrix}$ of the S matrix. However, we do not need to do this thanks to (b) (No new work is required!). Use the orthonormality of the two column vectors , and the fact that the S matrix should go to $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ as the well potential vanishes, to determine S_{12} and S_{22}.

5. [25 points; Coherent State, Part II] In Homework 4.5, we defined a coherent state for the Harmonic oscillator problem as an eigenstate of $\hat{a}_-$. The eigenvalue was given by λ, which can be any complex number in general. The eigenstate was given as a gaussian wave function [notation change: from $f(x)$ to $\Psi(x)$] $\Psi(x) = \sqrt[4]{\frac{2}{\pi A^2}} \exp(-\lambda_I^2) \exp\left[-\frac{(x-\lambda A)^2}{A^2}\right]$ where $A = \sqrt{\frac{2\hbar}{m\omega}}$. [Here λ_I is the imaginary part of λ. When λ is complex, the additional factor $\exp(-\lambda_I^2)$ included here is necessary for proper normalization.] This wave function satisfies the minimum uncertainty relation. We take $\Psi(x)$ to be our initial state wave function, and we use the Dirac notation $|\Psi(0)\rangle$, where 0 means $t = 0$, for that.

(a) Explain why λ needs to be complex to give a non-zero average momentum $\langle \hat{p} \rangle$. What is $\langle \hat{x} \rangle$? [You do not need to actually work out any integrals to answer these questions. All you need to do is to discuss generic features of the integrals.]

(b) Suppose that at $t = 0$, λ is a real number. If $\lambda = 0$, what is the nature of the state? If $\lambda \neq 0$, what is the nature of the state? [Hint: for $\lambda = 0$, it is sufficient to think about the eigenvalue equation for $\hat{a}_-$.]

(c) For this part, let us use the *Heisenberg picture* of quantum mechanics. Solve the Heisenberg equation of motion for $\hat{a}_-$ to show that $\hat{a}_-(t) = \hat{a}_-(0) \exp(-i\omega t)$.

(d) Now, come back to the Schrödinger picture. Show that the answer in (c) means that at any arbitrary time $|\Psi(t)\rangle$ is still an eigenstate of $\hat{a}_-$, but with eigenvalue $\lambda \exp(-i\omega t)$. [Hint: start from $\hat{a}_-(t)|\Psi(0)\rangle = \lambda(t)|\Psi(0)\rangle$ in the Heisenberg picture, and multiply on the left by $\hat{\mathbb{U}}(t)$, the time evolution operator, to convert it to an equation in the Schrödinger picture. Note that $\hat{a}_-(t) = \hat{\mathbb{U}}(t)^{-1}\hat{a}_-(0)\hat{\mathbb{U}}(t)$.]

(e) Starting at $t = 0$ from $\lambda =$ non-zero real number, describe, qualitatively, what happens to Δx, Δp, $\Delta x \Delta p$, $\langle \hat{x} \rangle$, $\langle \hat{p} \rangle$ as a function of time. The results of (a,b,d), the above wave function, the minimum uncertainty nature of that wave function for any λ, and the Ehrenfest theorem, $d\langle \hat{p} \rangle/dt = \langle -\partial V/\partial \hat{x} \rangle$ and $md\langle \hat{x} \rangle/dt = \langle \hat{p} \rangle$ should be useful.