

# Homework 4

Phys 139A, Spring 2008, UCSC

Due May 1, 5 pm

- [20 points; Linear Operators and Time Evolution Operator; Some really basic math and cool physics to remember at the end]
  - Prove that if  $\hat{A}$  and  $\hat{B}$  are linear operators, then  $\hat{A}\hat{B}$  and  $\hat{A} + \hat{B}$  are also linear operators. The definitions of product and sum operators are:  $\hat{A}\hat{B}\psi(x) = \hat{A}(\hat{B}\psi(x))$  and  $(\hat{A} + \hat{B})\psi(x) = \hat{A}\psi(x) + \hat{B}\psi(x)$ . [Hint: Recall that an operator  $\hat{O}$  is linear if  $\hat{O}[c_1\psi_1(x) + c_2\psi_2(x)] = c_1\hat{O}\psi_1(x) + c_2\hat{O}\psi_2(x)$  for any functions  $\psi_1(x)$ ,  $\psi_2(x)$  and any complex constants  $c_1$ ,  $c_2$ . In proving the linearity of  $\hat{A}\hat{B}$ , recognize that  $\hat{B}\psi_1(x)$  or  $\hat{B}\psi_2(x)$  is just another function of  $x$ .]
  - Show that  $\hat{H} = \hat{T} + \hat{V} = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x)$  is a linear operator. [This was demonstrated in class, when we said that if  $\Psi_1$  and  $\Psi_2$  are solutions for the time-dependent SE, then  $\alpha\Psi_1 + \beta\Psi_2$  is also a solution, so this problem is a re-cap! However, this is so important that I am not ashamed to ask again!]
  - Using (a,b), prove that any analytic function of  $\hat{H}$ ,  $F(\hat{H}) = \sum_{n=0}^{\infty} \frac{F^{(n)}(0)}{n!} \hat{H}^n$ , where  $F^{(n)}(0)$  is the  $n$ -th derivative of  $F$  evaluated at origin, is also a linear operator.
  - Using (c), show that  $F(\hat{H}) \sum_n c_n \psi_n(x) = \sum_n c_n F(E_n) \psi_n(x)$  where  $\psi_n(x)$ 's are stationary states with energy eigen-values  $E_n$ .
  - As we covered in class, THE general solution for the time dependent Schrödinger equation for a given initial condition  $\Psi(x, t = 0)$  is given by:

$$\Psi(x, t) = \sum_n c_n \psi_n(x) \exp(-iE_n t/\hbar)$$

where  $c_n = \int_{-\infty}^{\infty} dx \psi_n^*(x) \Psi(x, t = 0)$ . Prove, using (d) with  $F(\hat{H}) \equiv \hat{U}(t)$  below, that this general solution can be even more compactly expressed as

$$\Psi(x, t) = \exp(-i\hat{H}t/\hbar) \Psi(x, t = 0)$$

For an obvious reason, the operator  $\hat{U}(t) \equiv \exp(-i\hat{H}t/\hbar)$  is called a “time evolution operator.” [Note that  $\hat{U}(t)\Psi(x, t = 0)$  is simply the solution of the Schrödinger equation  $i\hbar\frac{\partial}{\partial t}\Psi(x, t) = \hat{H}\Psi(x, t)$ , assuming  $\hat{H}$  is not an explicit function of  $t$ . As discussed in class, this assumption is valid throughout this course.]

- [15 points] Problem 2.13 [Note: It is strongly recommended that the actual form of energy eigenfunctions  $\psi_n(x)$  *not* be used in this problem. When you write down  $\Psi(x, t)$  and  $|\Psi(x, t)|^2$ , simply leave  $\psi_0(x)$  and  $\psi_1(x)$  as symbols – do not write out the actual functions. Using equations [2.66-69], you should be able to obtain answers without knowing the actual forms of  $\psi_0$  and  $\psi_1$ .]

3. [10 points; hint – see page 14 of lecture note L5] Problem 2.41
4. [10 points] Problem 2.42
5. [15 points; Coherent State, Part I]

(a) Show that  $\hat{a}_{\pm}$  [in the  $x$ -representation] can be written as

$$\hat{a}_{\pm} = \frac{x}{A} \mp \frac{A}{2} \frac{\partial}{\partial x}$$

with  $A > 0$ . Express  $A$  in terms of  $m, \hbar, \omega$ . What is its dimension?

- (b) Prove that  $\hat{a}_{+}$  cannot have any normalizable eigenfunction. Namely, there is no such normalizable function  $f(x)$  that satisfies  $\hat{a}_{+}f(x) = \lambda f(x)$  where  $\lambda$  is a, generally complex, number. [Hint: you can prove this in two different ways. (i) You can express  $f(x) = \sum_n c_n \psi_n(x)$  where  $\psi_n(x)$ 's are eigenstates of the simple harmonic oscillator problem, using the completeness of  $\{\psi_n\}$ . Then, you can base your argument on what happens when you apply the operator  $\hat{a}_{+}$  on each  $\psi_n$ . (ii) You can try to solve the differential equation  $\hat{a}_{+}f(x) = \lambda f(x)$ , like in (c), and show that there is no normalizable solution.
- (c) Prove that  $\hat{a}_{-}$  does have normalizable eigenfunctions, by directly solving the differential equation  $\hat{a}_{-}f(x) = [\frac{x}{A} + \frac{A}{2} \frac{\partial}{\partial x}] f(x) = \lambda f(x)$ . Find the general form of the normalized eigenfunction. Such wave function represents the so-called “coherent state,” first discovered by Schrödinger. [Hint: define  $y = (x - \lambda A)^2$  and rewrite the differential equation in terms of  $y$ .]