§A.1 Wave function normalization and $\left.\frac{\partial^n \Psi}{\partial x^n}\right|_{\pm\infty}$

The normalization condition

$$\int_{-\infty}^{\infty} dx\ \Psi^*(x,t)\,\Psi(x,t) = 1$$

means that $\Psi(\pm\infty, t) = 0$.

How about $\frac{\partial \Psi}{\partial x}$ or $\frac{\partial^2 \Psi}{\partial x^2}$ etc. ?

Suppose $\frac{\partial \Psi}{\partial x} \longrightarrow$ const. for $x \to \infty$ (or $-\infty$)

Then, it means $\Psi \longrightarrow$ const. $\times$ x as $x \to \infty$
times
Then, Ψ cannot go to 0!

So, $\frac{\partial \Psi}{\partial x}$ has to go to 0 at $x = \pm\infty$.

Similarly for $\frac{\partial^2 \Psi}{\partial x^2}$ etc.

$$\left.\frac{\partial^n \Psi}{\partial x^n}\right|_{\pm\infty} = 0 \quad n = 0, 1, 2, \cdots$$

Note that when we did integration by parts quantities such as $\left.\Psi^* \frac{\partial \Psi}{\partial x}\right|_{-\infty}^{\infty}$ (section §1.4) or $\left.\frac{\partial \Psi}{\partial x}\frac{\partial \Psi^*}{\partial x}\right|_{-\infty}^{\infty}$ were treated as zeroes. ~~despite the fact that~~ This is why.

- Good news ··· In the matrix formulation (chap. 3), we do <u>not</u> have to deal with these things !! (not explicitly I mean.)

(A2)

§. What if $\psi(x) \underset{x\approx 0}{\sim} x^{\alpha}$ $\alpha < 0$?
$\psi(0) \to \infty$ or $-\infty$

To be normalizable
$$\int dx\, |\psi(x)|^2 = \text{finite}$$

Near $x=0$, $|\psi(x)|^2 \sim |x|^{2\alpha}$ $\to$ $2\alpha > -1$
$\therefore\ \alpha > -\frac{1}{2}$

$$\underbrace{-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi}_{\sim x^{\alpha-2}} + \underbrace{V(x)\psi}_{\sim x^{\alpha}} = \underbrace{E\psi}_{\sim x^{\alpha}}$$

(strongly divergent if $\alpha < 0$)

Assuming $|V(x)| < \infty$ for any finite x

$\Rightarrow$ Contradiction ! if $\alpha < 0$

[OK if $\alpha = 1$ or $\alpha = 0$. The 1st term $= 0$.]

$\Rightarrow$ $\psi(x)$ cannot be x^{α} near $x \approx 0$ with $\alpha < 0$ if $|V(x)| < \infty$ for any finite x

$\Rightarrow$ $\psi(x)$ should be finite.

[You can also show that $\psi(x) \sim \log x$ is not allowed either.]

It also follows that $\psi'(x)$ should be finite.