

Homework 4

Phys 155, Winter 2008, UCSC

Due Feb 7

Each problem is 5 points. For problems from H&H, please note that solutions are included in the textbook. However, use those solutions simply as a guide, if necessary. If you do, your answers should show clear evidence (your own words, logic, details, diagrams, etc.) for your own understanding.

1. H&H, 2.4; Be sure to demonstrate your mastery of every step of the derivation.
2. H&H, 2.6, with an addition: Using the Heisenberg uncertainty principle, estimate the uncertainty of the atomic position, Δx , associated with the zero point energy. Express Δx as a fraction of the Ar-Ar distance = 3.72Å.
3. [Alternative to using the “Sommerfeld expansion” for the free electron problem] Consider the density of states of electrons $g(\varepsilon)$. Note that at any temperature $\int_0^\infty g(\varepsilon)f(\varepsilon, \mu, T)d\varepsilon = N$, N = total number of electrons. Here f is the Fermi-Dirac distribution function, where for clarity all of its parameters are spelled out as functional arguments. As we covered in class, and as can be easily seen, at $T = 0$, this reduces to $\int_0^{\varepsilon_F} g(\varepsilon)d\varepsilon = N$, which defines the Fermi energy ε_F . Due to the Fermionic nature of electrons, when one electron is added or removed at $T = 0$, energy cost (gain) involved is ε_F (within $O(1/L)^2$), and so $\mu(T = 0) = \varepsilon_F$. The goal of this problem is to figure out the leading order finite temperature correction for $\mu(T)$, given a general form of the density of states. Note that, as usual, we work in the regime $T/T_F \ll 1$, and so the Taylor expansion near the chemical potential

$$g(\varepsilon) \approx g(\mu) + g'(\mu)(\varepsilon - \mu) \quad (1)$$

is expected to be useful. [Note that $\mu = \mu(T)$, always.]

- (a) The particle conservation means that $\int_0^\infty g(\varepsilon)f(\varepsilon, \mu, T)d\varepsilon = \int_0^{\varepsilon_F} g(\varepsilon)d\varepsilon (=N)$. Show that this equation leads to

$$\int_{\varepsilon_F}^{\mu} d\varepsilon g(\varepsilon) = \int_0^{\infty} d\varepsilon g(\varepsilon)[f(\varepsilon, \mu, T = 0) - f(\varepsilon, \mu, T)] \quad (2)$$

- (b) Now, let us assume that $\mu \approx \varepsilon_F$ to leading orders in T/T_F . This assumption will be justified soon (in (c)). Explain why the approximate eq. 1 can be used in eq. 2.
- (c) Use eq. 1 in eq. 2, and show (by iteration) that to leading order in T

$$\mu \approx \varepsilon_F - \frac{\pi^2}{6} \frac{g'(\varepsilon_F)}{g(\varepsilon_F)} (k_B T)^2 \quad (3)$$

[Hint: What is the parity of $f(\varepsilon, \mu, T = 0) - f(\varepsilon, \mu, T)$, when it is considered as a function of $\varepsilon - \mu$? Also, use $\int_0^\infty \frac{xdx}{\exp(x)+1} = \frac{\pi^2}{12}$.]

- (d) Show that eq. 3 gives $\mu \approx \varepsilon_F \left[1 - \frac{\pi^2}{12} \left(\frac{T}{T_F} \right)^2 \right]$ for free electrons in three dimensions, as discussed in class (slide 6 of lecture 8). What is the corresponding answer for μ in two dimensions?
- (e) Show that in three dimensions, $g(\varepsilon_F) = \frac{3}{2} \frac{N}{\varepsilon_F}$ and $E(T=0) = \int_0^{\varepsilon_F} d\varepsilon \varepsilon g(\varepsilon) = \frac{3}{5} N \varepsilon_F$.
- (f) [Bonus 5 points, if you do this optional problem] For the total energy $E = \int_0^\infty d\varepsilon \varepsilon g(\varepsilon) f(\varepsilon, \mu, T)$ show that, by employing the above techniques (splitting f as in (a) and noting the symmetry of $f(\varepsilon, \mu, T=0) - f(\varepsilon, \mu, T)$ as in (c), etc.), to the leading order $E \approx E(T=0) + \frac{\pi^2}{6} (k_B T)^2 g(\varepsilon_F)$. For three dimensions, show that $E \approx \frac{3}{5} N \varepsilon_F + \frac{\pi^2}{6} (k_B T)^2 g(\varepsilon_F) = \frac{3}{5} N \varepsilon_F + \frac{\pi^2}{4} N \varepsilon_F \left(\frac{T}{T_F} \right)^2$ (results of (e) would be helpful; this result was stated in slide 6 of lecture 8), and thus $C_V \approx \frac{\pi^2}{2} N \frac{T}{T_F} k_B$ (the result stated in slide 9 of lecture 8). [It is again important to see that $E - E(T=0) \approx AT^2$ no matter what – dimension or dispersion relation – as long as $T/T_F \ll 1$ and $g(\varepsilon_F)$ is finite.]