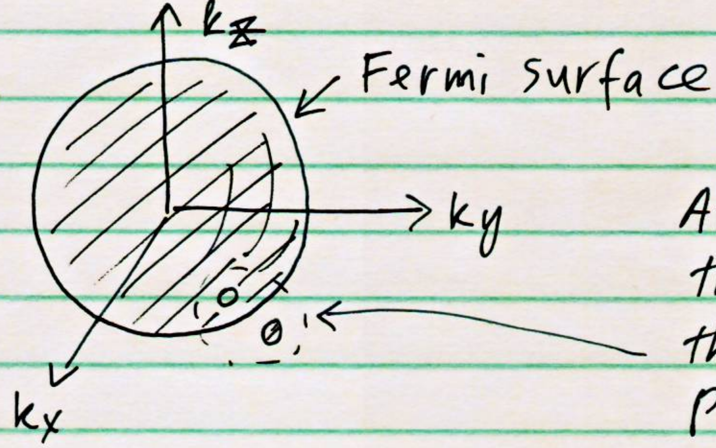
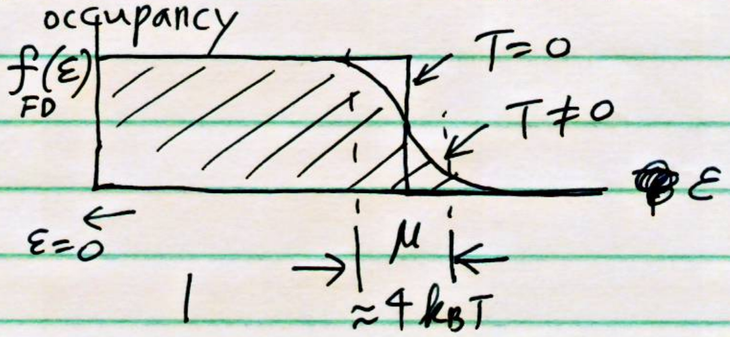
Why $C_V \propto T$ for electron gas



At finite temperature the system develops these electron-hole pairs.

The mechanism by which these electron-hole pairs appear is the Fermi-Dirac distribution function



$$f_{FD}(\varepsilon) = \frac{1}{e^{(\varepsilon-\mu)/k_BT}+1}$$

$f(\varepsilon)$ is the occupancy of the orbital with energy ε.

$f(\varepsilon)$ is different from a step function ($T=0$) only near μ.

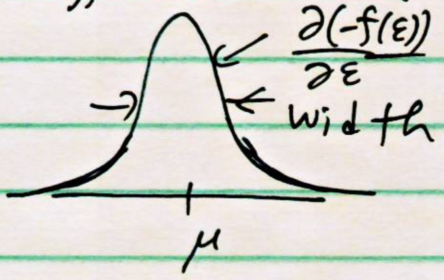
More quantitatively $f(\varepsilon)$ is different from a step function in energy range

$$[\mu - 2k_BT,\ \mu + 2k_BT].$$

That is we can say fairly accurately that

$$f(\varepsilon > \mu + 2k_BT) \approx 0, \quad f(\varepsilon < \mu - 2k_BT) \approx 1$$

(Another way of saying this is that if you differentiate $(-f(\varepsilon))$ then you get

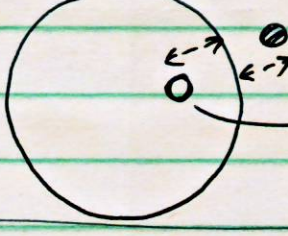


)

In any case, the number of electron-hole pairs generated is about (for $k_BT \ll \varepsilon_F$)

$$N_{e-ph} \approx g(\mu)\cdot(2k_BT)\cdot\frac{1}{4}$$

The energy of an electron-hole pair is,

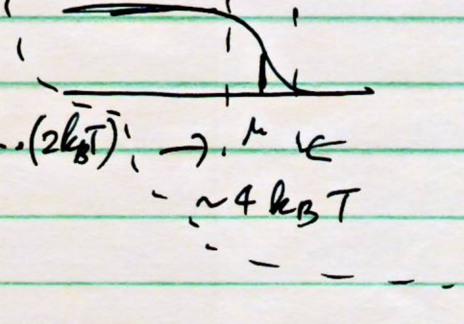


add electron energy increase ~~cost~~ $\approx \mu + k_BT$

remove electron energy ~~gain~~ decrease $\approx \mu - k_BT$

because this is the typical distance in energy axis of electron or hole again, due to f_{FD}

Total energy increase $\approx 2k_BT$!



Thus $E - E(T=0) \approx g(\mu)\cdot(2k_BT)\cdot\frac{1}{4}\cdot(2k_BT)$

$\approx g(\mu)(k_BT)^2$

We "know" that $\mu \approx \varepsilon_F + O(T^2)$
(see ↑ next note)

So, $E - E(T=0) \approx g(\varepsilon_F)(k_B T)^2$

↑

This expression is wrong
by a factor of $\pi^2/6$

Compared to the 3D free e^- gas

However, notice that in the above discussion this behavior

$$E - E(T=0) \propto T^2$$

is true for any dimension
any dispersion relation
as long as $\varepsilon_F \gg T$

$$\boxed{C_V = \left(\frac{\partial E}{\partial T}\right)_V \sim g(\varepsilon_F)\cdot(k_B T)\cdot k_B}$$

Why $\mu \approx \varepsilon_F + O(T^2)$ for e^- gas

We did not really prove this. While Sommerfeld expansion is the way to prove this mathematically, let's show this using qualitative arguments.

① It's easy to see that $\mu \approx \varepsilon_F$.

~~Since~~

~~The~~ μ-defining equation is for any T

$$\int_0^\infty d\varepsilon\, g(\varepsilon)\, f_{FD}(\varepsilon) = N \qquad \text{---}(*)$$

Basically what happens is that there are electrons above μ and holes below μ at finite T.

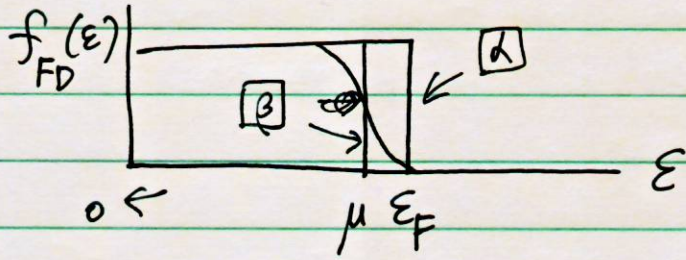
$$\#\text{ of electrons} \approx g(\mu)\,(k_BT)/2 \qquad \ldots(**)$$

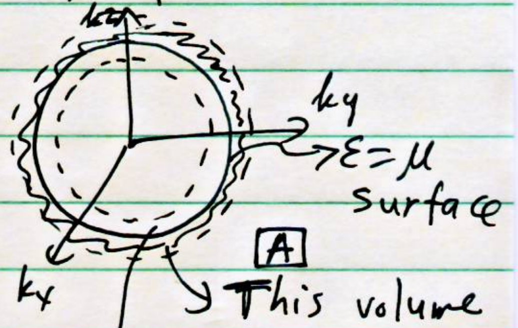
$$\#\text{ of holes} \approx g(\mu)\cdot\frac{(k_BT)}{2} \qquad \text{---}(***)$$

In this approximation, there is no reason why μ has to depart from ε_F since $(**)$ and $(***)$ are the same when

$$\mu = \varepsilon_F \qquad \therefore \mu \approx \varepsilon_F$$

② Now let's be somewhat more sophisticated.
What really happens is that (in 3D)





$\boxed{A}$ This volume $\propto$ # of e^-s relative to step function at μ $\boxed{\beta}$

$\boxed{B}$ This volume $\propto$ # of h's relative to step ftn at μ $\boxed{\beta}$

In 3D, $\boxed{A} > \boxed{B}$ (because $g(\varepsilon)$ increases as $\varepsilon\uparrow$)
that is μ will become less than ε_F in order to keep the number of e^-'s the same as $T=0$ value.

The equation for determining μ for any D, dispersion

$$\Rightarrow \quad g(\varepsilon_F)(\mu-\varepsilon_F)+c\,g(\varepsilon_F+k_BT)(k_BT) - c\,g(\varepsilon_F-k_BT)(k_BT)=0$$

1st term : Change of the # of e^-'s for step ftn at μ rel. to step ftn at ε_F
$\boxed{\beta}$ $\boxed{\alpha}$

2nd term : Change of the # of e^-'s for $f_{FD}(\varepsilon)$ relative to step ftn at μ

3rd term : – Change of the # of h's for $f_{FD}(\varepsilon)$ relative to step ftn at μ

C : # of order 1.

This equation gives

$$g(\varepsilon_F)\cdot(\mu-\varepsilon_F)+2C\,g'(\varepsilon_F)(k_BT)^2=0$$

$$\therefore\ \mu=\varepsilon_F-2C\cdot\frac{g'(\varepsilon_F)}{g(\varepsilon_F)}(k_BT)^2$$

I.e. $\mu=\varepsilon_F+O(T^2)$

Note that in 3D

μ decreases → we know this from above

in 2D

μ does not change → because $g'(\varepsilon)=0$

in 1D

μ increases ~~at~~ ~~fini~~ → reasonable.

at finite T