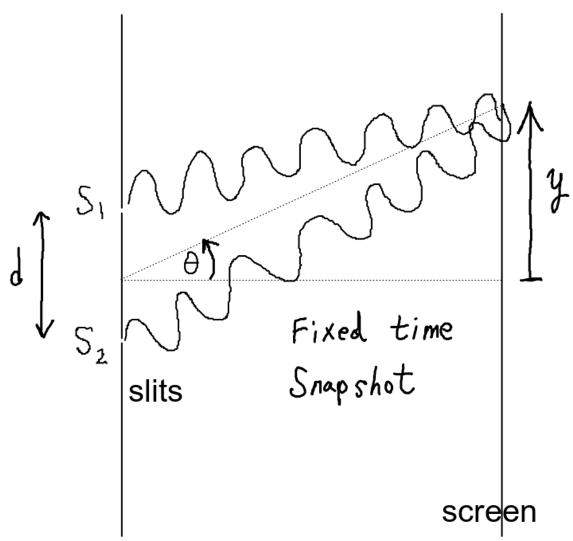
Notes for Lecture 12

Young's double slit experiment

Again, you must read the book and your lecture notes taken during the lecture to go over all basic materials. Here, we present a more rigorous derivation of Young's double slit formulae.

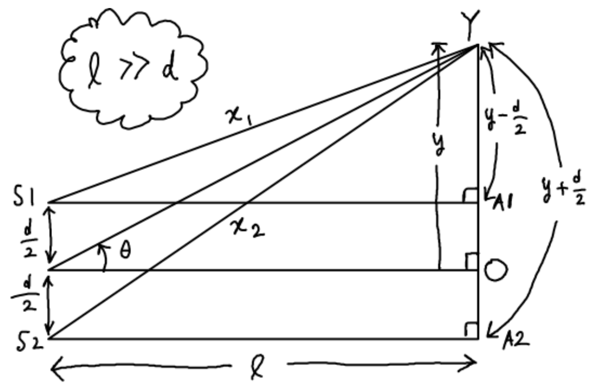
The diagram below summarizes the basic setup.



The two narrow slits S1 and S2 define two point sources of coherent light. Spherical waves of light emanate from these two points. When these two spherical waves interfere, they form various minima and maxima in intensity. This interference pattern can be observed by placing a screen as shown and measuring the intensity of light there.

Note that this problem is very similar to the sound interference problem involving two speakers.

In order to treat this problem mathematically, let us consider the following diagram.



An important assumption[1] is that $l \gg d$. Under this single assumption, the following conditions can be derived.

$$d \sin\theta = m\lambda \qquad \text{constructive interference} \tag{12.1}$$

$$d \sin\theta = \left(m + \frac{1}{2}\right)\lambda \qquad \text{destructive interference} \tag{12.2}$$

$$m = 0, \pm 1, \pm 2, \ldots \tag{12.3}$$

The bright light one gets due to the $m = 0$ constructive interference condition is called the zero-th order light. The bright light one gets due to the $m = \pm 1$ constructive interference condition is called the first order light. And similarly for the second order light and higher order lights.

Let us derive the above conditions from the elementary considerations of two spherical waves interfering. The following derivation is *not* required, but optional for you to learn. The wave amplitude[2] at y is given by

$$D(y,t) = D(x_1,t) + D(x_2,t) \tag{12.4}$$

$$= \frac{A}{x_1}\sin(kx_1 - \omega t + \phi) + \frac{A}{x_2}\sin(kx_2 - \omega t + \phi) \tag{12.5}$$

$$\approx \frac{A}{x_1}\sin(kx_1 - \omega t + \phi) + \frac{A}{x_2}\sin(kx_1 - \omega t + \phi + k\Delta x) \tag{12.6}$$

[1] This type of assumption is routinely made in a "scattering experiment." Also, see footnote 3.

[2] This usage of the word "amplitude" is common and should not be confused with A as amplitude. One often refers to the whole solution to a wave equation, such as $D(y,t)$ here, as "amplitude."

where $\Delta x \equiv x_2 - x_1$. So far, our expressions have been very general, and apply even when l is very small compared with d. However, now, we apply our assumption $l \gg d$. If this assumption is made, it is convenient to define

$$r \equiv \sqrt{l^2 + y^2} \tag{12.7}$$

From the above diagram, $x_1 = \sqrt{l^2 + \left(y - \frac{d}{2}\right)^2}$. Clearly, $x_1 \approx r$. We can do better, by evaluating the first order term in d using the Taylor expansion theory of calculus. We can do similarly for x_2. We get[3]

$$x_1 = r - \frac{yd}{2r} + O(d^2) \tag{12.8}$$

$$x_2 = r + \frac{yd}{2r} + O(d^2) \tag{12.9}$$

Plugging these results into the expression for $D(y,t)$, we get

$$D(y,t) \approx \frac{A}{r}\left(\sin X + \sin(X + \delta)\right) \tag{12.10}$$

$$X \equiv kx_1 - \omega t + \phi \qquad \text{common phase} \tag{12.11}$$

$$\delta \equiv k\Delta x = k(x_2 - x_1) \approx \frac{kyd}{r} = kd\sin\theta \qquad \text{phase difference} \tag{12.12}$$

Clearly, we get a constructive interference if the phase difference δ is 2π times an integer, and a destructive interference if δ is an odd integer times π, since $\sin(X+\pi) = -\sin X$. Thus, we get

$$k\Delta x = 2\pi m \approx kd\sin\theta \Leftrightarrow d\sin\theta = m\lambda \tag{12.13}$$

$$k\Delta x = 2\pi\left(m + \frac{1}{2}\right) \approx kd\sin\theta \Leftrightarrow d\sin\theta = \left(m + \frac{1}{2}\right)\lambda \tag{12.14}$$

proving what we set out to prove.

[3] Since the $O(d^2)$ term is of the form d^2/r^2 up to a constant, there exists an *additional* requirement if we were to ignore that term as we do here: $|y| \gg d$. So, both $|y = r\sin\theta|$ and r must be much greater than d. Clearly, as $r \to \infty$, we see that both quantities will be much greater than d assuming θ is non-zero, but there is a subtle important point to be made here. Note that in our result here the smallest non-zero significant value of $|\sin\theta| \sim \frac{\lambda}{d}$. And so, $|y| \gg d$ means $r\lambda/d^2 \gg 1$. Or, since $r \sim l$, we get $l\lambda/d^2 \gg 1$. This condition is the so-called Fraunhofer diffraction condition (for far-field diffraction) as opposed to the so-called Fresnel diffraction condition (for near-field diffraction). The number $d^2/(l\lambda)$ defines the so-called Fresnel number. These discussions also apply to the case of a single slit as well (see later lecture); in that case, d must be replaced by D, the linear dimension of the single slit. In any case, our focus in this course is on the Fraunhofer case only; that is, our underlying assumption is that the Fresnel number is small.