Notes for Lecture 4

Waves

Here we go now. After we studied one of the most fundamental problems in physics—SHO, now we take on one of the most important phenomena in physics—waves. Not surprisingly, soon we will see that SHO and wave are two tightly connected phenomena.

4.1 Wave—what is it?

Waves are everywhere. But, what are waves? This is generally *not* an easy question to answer. There is a certain mystery about the wave phenomena in general, except classical mechanical waves. In fact, we do not understand waves such as light waves very much. Einstein famously quipped "for the rest of my life, I will contemplate on what light is." Imagine this coming from Einstein!

So much about what we don't understand. Let us consider what we *do* understand—mechanical waves. The defining characteristics of mechanical waves are given in a box below.

Examine Figure T15-1 with these characteristics in mind. First, note that each small segment of a string can be considered a particle. So, a string consists of many particles, confirming characteristic 1. Second, the wave propagation velocity and the velocity of each particle in the string are quite different. Indeed for this example of a string wave, the average velocity of each particle is zero, while the wave itself has a definitely finite velocity, even when averaged over time. This confirms characteristic 2. Lastly, note that a small segment of the string will not participate in a wave, unless its adjacent segment happens to move and thus exerts a force on the original segment. This confirms characteristic 3.

Perhaps even more understandable than a string wave is a "cheer wave" in a baseball stadium. Please verify all the above characteristics for the cheer wave. Especially, think of the last characteristics. When a cheer wave is going around in a baseball stadium, it does *not* mean that everyone is cheerful or even paying attention to the baseball game!

Mechanical wave

1. **Wave is a collective phenomenon.** It involves many particles, not just one particle.
2. **Wave moves far, but participating particles do not.** Wave propagates, moving energy, but each constituent particle stays at a fixed position on the average. What propagates as a wave is not constituent particles themselves but the disturbance of each constituent particle around its fixed average position.
3. **Wave is an emergent phenomenon.** Wave emerges due to the interaction of particles, and goes beyond the properties of constituent particles. Typically, wave is caused by the local interaction of a constituent particle with its "contact neighbors" only. Through typically local interactions, a wave phenomenon happens, and represents not the property of one or a few constituent particles, but the emergent property of *all* particles.

4.2 Travelling sinusoidal wave

Not all waves are **sinusoidal** (or, in another word, **harmonic**). Nevertheless, it is very convenient to consider sinusoidal waves, first. The reason? Due to the Fourier theorem, pretty much all wave forms can be expressed as a linear combination of sinusoidal waves.

A travelling sinusoidal wave can be written down as

$$D(x,t) = A\sin(kx - \omega t + \phi) \qquad A \geq 0, \quad \omega \geq 0 \tag{4.1}$$

This is a right moving sinusoidal wave, and soon we will get to the left moving one. Here, $D(x,t)$ means the local disturbance/displacement for the particle at x, at time t. For a wave on a string (like a guitar string), D is the displacement *perpendicular* to the line defined by the string line in equilibrium. Such a wave is called a **transverse** wave. For sound wave propagating through any medium, D can be taken as the average displacement of local atoms. Sound wave in air is a so-called **longitudinal** wave: the disturbance is parallel (or anti-parallel) to the wave propagation direction.

Note that above, we assumed $A \geq 0$ and $\omega \geq 0$, without loss of generality. The angular frequency (ω) is always positive or zero. Should you encounter a wave expression like the above but with a negative value of A, then you can always shift the phase by π, and turn the amplitude positive. Here is what I mean. Suppose somebody gave you a wave expression: $D(x,t) = A\sin(kx - \omega t + \phi)$ where $A < 0$. Such expression does not fit our definition above. However, let us define $\phi' \equiv \pi + \phi$ as the new initial phase. Since $-\sin(z + \phi) = \sin(z + \phi + \pi) = \sin(z + \phi')$ where $z \equiv kx - \omega t$, we get $D(x,t) = (-A)\sin(kx - \omega t + \phi') = A'\sin(kx - \omega t + \phi')$, which fits our definition above, now with a positive amplitude $A' = -A$ and a new initial phase $\phi' = \phi + \pi$.

In general, we do *not* need to make any assumption about the sign of k (see the box in page 5), but, for the purpose of *this* course, we shall do so for the most part: we shall use the convention $k > 0$, and will take note if we go beyond this convention.

D is a function of two variables, and so we will have to examine it carefully, and will have to use the (basic) multi-variable calculus at some point.

For a fixed x value, D represents a sine function of time. Thus, each particle at each value of x goes through a simple harmonic motion! This is why the above wave is alternatively called a **harmonic wave**. The relation

$$\omega = \frac{2\pi}{T} = 2\pi f \qquad \text{angular frequency, period, frequency} \tag{2.1–3}$$

remains valid without any need for modification.

Note that for a fixed t value, $D(x,t)$ is a sine function of x. Its periodicity in x is called **wavelength**:

$$\lambda = \frac{2\pi}{k} \qquad \textbf{wavelength (lambda)} \tag{4.2}$$

$$k = \frac{2\pi}{\lambda} \qquad \textbf{wave number} \tag{4.3}$$

The above sinusoidal wave is a right-moving harmonic wave. Why? To know how the above wave form moves, it is sufficient to pick one point on the wave form. We can do so by taking

$$kx - \omega t + \phi = constant \tag{4.4}$$

where *constant* can be, e.g., $\pi/2$ (a top point of a sine wave), 0 (a mid point) or $3\pi/2$ (a bottom point). Whatever this particular point of the wave form we pick, the above equation describes how its x and t coordinates change. By taking the differential on both sides, we can figure out how our chosen point moves

$$kdx - \omega dt = 0 \tag{4.5}$$

$$\therefore\ v \equiv \frac{dx}{dt} = \frac{\omega}{k} \qquad \text{wave velocity} \tag{4.6}$$

By the same reasoning, we can state the following.

$$D(x,t) = A\sin(kx - \omega t + \phi), \qquad \text{right moving harmonic wave} \tag{4.7}$$

$$D(x,t) = A\sin(kx + \omega t + \phi), \qquad \text{left moving harmonic wave} \tag{4.8}$$

$$D(x,t) = g(x - vt), \qquad \text{right } (v > 0) \text{ or left } (v < 0) \text{ moving general wave} \tag{4.9}$$

where g is an arbitrary function. Note that a general wave can also be written as $g(kx \pm \omega t)$. However, that form is not nice, since it can be misleading; k can be interpreted as wave number *only if* g is sinusoidal (or periodic, at the very least; but, see Section 4.4).

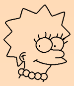
What happens if $k = 0$? This corresponds to an infinite wave length. As even the size of the Universe is not infinite, to our knowledge, the $k = 0$ case need not be considered. Having said this, note that the $k = 0$ case is quite often discussed in physics. What is really meant is the case when λ is really large, much larger than all length scales of a given problem.

4.3 Velocities

In the above we have already identified ω/k (or $\omega\,\hat{k}/|\vec{k}|$) as the velocity of the wave. This makes sense, since it means

$$v = \frac{\omega}{k} = \frac{\lambda}{T} \qquad \textbf{wave velocity} \tag{4.10}$$

where the two completely analogous relations, $\omega = 2\pi/T$ (T represents the periodicity in time) and $k = 2\pi/\lambda$ (Eq. 4.3; λ represents the periodicity in space), are used.



Wave vector

You can skip reading the content of this box, without much adverse effect.
However, the content is of general importance.

A much more general way of viewing k is to view it as a **wave vector**, not as a wave number. For a one dimensional wave, this can be accomplished by removing the restriction on the sign of k. For waves in higher, or general, dimensions, we can write

$$D(\vec{x},t) = A\sin(\vec{k}\cdot\vec{x} - \omega t + \phi) \tag{4.11}$$

$$\lambda = 2\pi/|\vec{k}| \qquad |\vec{k}| \text{ is the wave number} \tag{4.12}$$

The velocity of this wave is given by

$$v = \frac{\omega}{|\vec{k}|}\hat{k} \tag{4.13}$$

where $\hat{k}$ is the unit length vector given by $\vec{k}/|\vec{k}|$, i.e., the unit vector in the direction of $\vec{k}$. For one dimensional wave, this means that we can write

$$D(x,t) = A\sin(kx - \omega t + \phi) \tag{4.14}$$

$$v = \frac{\omega}{k} \tag{4.15}$$

where v can be positive or negative depending on the sign of k.

4.4 An important convention

The quantity k (wavenumber or wave vector) could in principle associated with any periodic function, but that would make matters quite confusing. So, by convention, physicists use k only for sinusoidal waves.

As we shall see later, a sinusoidal wave can be a *standing* wave, as well as a *travelling* wave. The equation that we used above, $D(x,t) = A\sin(kx - \omega t + \phi)$, is a *travelling* sinusoidal/harmonic wave, for an obvious reason.

The wave velocity equation, Eq. 4.10, is then applicable **only for travelling sinusoidal waves**.

Phase velocity, Group velocity

You can skip reading the content of this box, without much adverse effect.
However, the content is of general importance.

Two different wave velocities are to be differentiated.

$$v_p = \frac{\omega}{k} \qquad \textbf{phase velocity} \tag{4.16}$$

$$v_g = \frac{d\omega}{dk} \qquad \textbf{group velocity} \tag{4.17}$$

The phase velocity is what we derived above. If you look at the derivation carefully, you will see that it is the velocity of a *single harmonic wave* with a definite frequency value. That is, **the phase velocity describes the wave velocity of a perfectly monochromatic wave**. Alas, such a wave does not exist. A **wave packet** is a more realistic form of wave. Of particular interest is a *nearly monochromatic wave*: a wave packet with a very small $\Delta\omega$, the width in ω. **A nearly monochromatic wave propagates at the group velocity, not at the phase velocity**. For this reason, generally, the group velocity is the more important physical quantity!

Note that v_p and v_g differ from each other only if the medium is **dispersive**. That is, if the phase velocity is dependent on k so that ω is not a linear function of k. The functional relation $\omega = \omega(k)$ is, in general, referred to as the **dispersion relation**. For the most part, in this course, we will deal with non-dispersive cases only, i.e. $v_p = \omega/k = \text{constant} = v_g$. However, we will encounter some important dispersion physics, such as rainbow or prism. Indeed, in general, one should not assume that $\omega/k = \text{constant}$.

4.5 Transverse and longitudinal waves

If the displacement $D(x, t)$ is perpendicular to the direction of the wave propagation, then it is a **transverse** wave. If the displacement $D(x, t)$ is parallel to the direction of the wave propagation, then it is a **longitudinal** wave.

The sound wave propagating in gas or liquid is a longitudinal wave. The string wave (Figure T15-1). The same medium can support both types of waves. For instance, the sound wave propagating in a solid can be longitudinal or transverse. So is a earthquake—a seismic wave. A slinky can support a transverse wave as well as a longitudinal wave.

In the above definition, the precise meaning of the "direction of the wave propagation" is the "direction of the wave vector."

4.6 What is intrinsic to the medium?

In the above, we learned that

$$\omega = vk. \tag{4.18}$$

This equation is just another form of Eq. 4.10.

A question—which of the three quantities appearing above is intrinsic to the medium through which the wave propagates? The answer is v. Why so? Let us consider a "banana slug observation wave," which is very analogous to a string wave or a baseball game cheer wave.

The set-up for the **banana slug observation wave** is this. A group of people form a straight line. Each person is able to see only the back of the person in front of her/him. The person at the beginning of the row is able to see a tree in front of her/him, and on that tree a banana slug is crawling. This first person's head follows the lateral movement of the banana slug (but not the vertical motion just to make the setup simple). As this person's head moves, the person behind copies the motion, which is copied by the next person and so on. Only the first person sees the banana slug, as every other person's attention is fixed at the back of the head of the person right in front of that person.

So, this example is quite analogous to the wave on string. Except the shaking of string is replaced by the banana slug crawling, the tension on the string is replaced by the human interaction. It is clear that this "banana slug wave" will propagate at a velocity that depends on how fast a typical person reacts to signal (a few hundred milli seconds) and the spacing between people in the row. Note that this velocity is determined by the properties of the row of people alone, and is independent of the banana slug, assuming that the banana slug is moving slow enough.

4.7 Wave velocity examples

Coming back to the wave on string, the velocity of the wave is given by

$$v = \sqrt{\frac{F_T}{\mu}} \tag{4.19}$$

where μ is the linear density of mass (mass per unit length), F_T is the tension on string.

This formula can be derived using a "long wave length" approximation. This approximation amounts to assuming that the wave amplitude is much smaller than the wave length. Or, $|v'| \ll |v|$, where v' is the velocity at which each segment moves up or down, and v is velocity at which the wave propagates. Note that for a travelling sinusoidal wave, $v = \lambda f$ and $|v'|_{max} = A\omega = A2\pi f$. Therefore, $|v'/v| \leq 2\pi A/\lambda$, which will become very small, indeed, in the long wave length limit.

In the long wave length limit, the tension on string is essentially unchanged from the tension on string in equilibrium. Using this, a simple derivation for v can be made, as shown during the lecture and as also shown in the textbook (from a slightly different point of view). However, you are advised to take the setting up of the wave equation (Section T15-5) as the best method to prove the wave velocity. So, an advanced student, who is comfortable with multi-variable calculus, is strongly encouraged to read T15-5 and understand it. In future lectures, we may or may not derive the wave equation, but we will **mention and use the wave equation** and discuss its properties (such as superposition).

The above expression for wave velocity, Eq. 4.19, describes the wave velocity for a particular transverse wave.

It is interesting to note that many wave velocities have the expression of the form $\sqrt{E/I}$, where E stands for the elastic property (not energy) and I stands for the inertial property (not the rotational inertia). You will note that this is also qualitatively true even for the expressions for ω's that we derived for typical SHO problems! This is not surprising at all, given Eq. 4.10. Another name for the "elastic tendency" is the "resilience" or the "restoring tendency"—i.e., the elastic property is closely connected to Hooke's law for small amplitudes.

The following two are other common examples for the wave velocity, both for longitudinal waves.

$$v = \sqrt{\frac{B}{\rho}} \qquad \text{sound wave in gas/liquid} \tag{4.20}$$

$$v = \sqrt{\frac{Y}{\rho}} \qquad \text{sound wave solid, longitudinal mode} \tag{4.21}$$

Here, ρ is the volume mass density, i.e., mass per unit volume. B is the bulk modulus, defined by $-B\Delta V/V \equiv \Delta P$. ΔP is a small change of the pressure of the system, and ΔV is a small change of the volume of the system, from V. Since an increase in pressure results in a decrease of volume, there is a minus sign in the defining equation for B so that B is positive. Note that B has the same physical dimension as P. Y is Young's modulus, $Y \equiv (F_{applied}/A)/(\Delta l/l)$, where $F_{applied}$ is the applied force on a solid of surface area A (and so $F_{applied}/A$ is the pressure in this case), l is the length of the solid parallel to $F_{applied}$, and Δl is the change of l in the direction of $F_{applied}$. It is important to note that both B and Y measure the resilience or the strength of the system in the sense that they measure the tendency for the system to resist the external pressure. The different definitions B and Y are required for gas/liquid and solid, since the property of a solid is not isotropic, i.e., its property may depend on the direction. Also, a solid can support a shear wave as well—a transverse sound wave.

Finally, let us note that the light is a wave, and it propagates at the speed of light, c, in vacuum. We do not yet know what elastic property and what inertial property, if any[1], of vacuum give rise to this fundamental constant of Nature. However, for now, it would seem to be the best practice not to get lost in thoughts about this perplexing mystery. It would seem to be the best student practice just to accept c as a fundamental constant.

[1] You might ask how can a vacuum have any property? In advanced courses, you will learn the following. In the modern view of physics, the vacuum is far from nothing. It has a very rich structure!