Notes for Lecture 3

Simple harmonic oscillation, cont.

3.1 SHM, UCM, and phase space

A uniform circular motion (UCM) and a simple harmonic motion (SHM) are deeply related. As explained in the last lecture note, a SHM can be thought of as a projection of a UCM (in an xy plane) onto the x axis (or any axis).

Consider a SHM such as the motion of a mass on spring. What UCM do are we talking about? In this case, all that is happening is a motion of the mass in the x axis, assuming that we measure the elongation or the compression of the spring using the x coordinate system.

That there *is* a UCM associated with *any* SHM can be seen by looking at the energy conservation equation, Eq. 2.16.

$$E = K + U = \frac{1}{2}mv^2 + \frac{1}{2}kx^2, \tag{2.16}$$

where E is constant, due to the mechanical energy conservation. This can be re-expressed as

$$\frac{2E}{k} = x^2 + \frac{m}{k}v^2 = x^2 + \left(\frac{v}{\omega}\right)^2 \qquad \omega \equiv \sqrt{\frac{k}{m}} \text{ (Eq. 1.9)} \tag{3.1}$$

$$= x^2 + \left(\frac{p}{m\omega}\right)^2. \qquad p \equiv mv\text{, the momentum} \tag{3.2}$$

Note that we got ourselves a circle equation, if we define

$$y \equiv -\frac{p}{m\omega} = -\frac{v}{\omega}. \qquad \text{Warning: this } y \text{ is } \textit{not} \text{ a coordinate in real space.} \tag{3.3}$$

Besides, note that, if we take $x = A\cos(\omega t + \phi)$, then $y = -\frac{p}{m\omega} = -\frac{v}{\omega} = A\sin(\omega t + \phi)$. Thus, we not only have a circle in the (x, y) plane, the motion on the circle is a UCM with constant angular velocity ω (see Section 2.5)! This (x, y) plane, where y corresponds to the (scaled) momentum corresponding to x, is the so-called **phase space**[1]. Now, you should feel much better about why we call $\omega t + \phi$ the phase (i.e., what does it have to do with the phase of the Moon, which obviously arises from the angular position in a circular motion; cf., Section 2.5)! In this view, ϕ determines the initial angular position on the circle.

Phase space and complex plane

Some physicists think that the complex plane is one of the greatest inventions of mathematics, and the complex plane is more real than the real axis in the complex plane. Read the following points, and see if you might agree (already—more use of complex numbers will be mandatory as you go to higher level classes; modern physics tell us that there is no escaping of physics from complex numbers).

1. The above phase space, (x, y) can in fact be viewed as the complex plane, if we define the complex number as usual: $z = x + iy$, with $i \equiv \sqrt{-1}$.

2. Due to Euler's formula,

$$e^{i\theta} = \cos\theta + i\sin\theta, \tag{3.4}$$

one can then represent the SHM as a single complex number

$$z = Ae^{i(\omega t+\phi)} \qquad A, \omega \geq 0. \tag{3.5}$$

As well known from the theory of complex numbers, A represents the radius, and $\theta = \omega t + \phi$ is the angle measured from the positive x axis.

[1] This is an example of phase space. In general, for a particle motion in n dimensions, the phase space has $2n$ dimensions, since we must associate a respective momentum axis to each real space axis.

3.2 SHO examples

The SHM is the basic tool to describe the wave phenomena. Before we start discussing waves, two important examples of the SHO are worth our attention. One is a pendulum (simple or physical), and the other is a torsion pendulum. We start with the latter.

3.2.1 Torsion pendulum

Let us take the torsion pendulum, first, since it is easier. As Figure T14-18 shows, the idea is that you have a thin wire of some sort (e.g., metal) on which a mass with the rotational inertia I is hanging. This mass and the wire are welded together, or firmly joined in some other way, so that when the mass is twisted, the wire itself gets twisted. The wire resists this, exerting a **restoring torque** of Hooke's law form, but in the angular coordinate, as in $\tau = -\kappa\theta$, where τ (tau) is torque, κ (kappa) is torsion coefficient, and angle θ is small

$$|\theta| \ll 1 \qquad \text{small angle (rad), e.g., 0.1 rad} = 6^\circ \tag{3.6}$$

Note that we use radian for the unit of angle, unless otherwise stated, in this course. The rotational form of Newton's second law then reads

$$I\ddot{\theta} = -\kappa\theta \qquad \text{Newton's 2nd law; } \kappa = K \text{ of the textbook} \tag{3.7}$$

This equation can be re-written as

$$\ddot{\theta} = -\omega^2\theta \qquad \omega \equiv \sqrt{\kappa/I} \tag{3.8}$$

Comparing this with Eq. 1.12, the fundamental EOM for SHO, we realize that this EOM is just another SHO EOM, *with θ replacing x*. Thus, the resulting motion is described by

$$\theta = A\cos(\omega t + \phi). \tag{3.9}$$

As we shall see shortly, this equation also describes the motion for a simple pendulum and a physical pendulum.

Before we go on further, one must realize that A has the dimension of the angle (dimensionless), i.e., the angular displacement, while in the previous problem of mass on spring it had the dimension of the displacement; they are not the same thing obviously.

Which angle?

Rotational SHM, like the one that we just identified for a pendulum, can be very confusing, since there are two angles involved. One is the real space angle, θ, and the other is the phase $\omega t + \phi$. **Do not mix them up!** Note that in the previous lecture note we used the notation $\dot{\theta} = \omega$ in Section 2.5, which was fine for the mass on spring problem, *but not for pendulum problems*, if, as we usually do, we use θ to represent the real angle, not the phase space angle.

If all this is confusing, I do not blame you. However, it is very important to make it clear what space you are considering. If nothing is confusing so far, then excellent. If all this seems to complicated, then follow this guideline for now. If a problem involves an oscillation (SHM) in real space angle, θ, *do not even think about the UCM* that is related to the SHM in a way described in Section 3.1!

Phase space for angular SHM?

As you might have guessed, it must be defined as a plane for which $(x, y) \equiv (\theta, -\frac{L}{I\omega})$. Here, L is the *angular* momentum: $L = I\dot{\theta}$, and I is the rotational inertia. $\omega = \sqrt{\frac{\kappa}{I}}$ (see above) or $\sqrt{\frac{g}{l}}$ (see below), and *not* $\dot{\theta}$!

3.2.2 Simple pendulum, physical pendulum

Figure 3.1 defines these two important pendulums. A simple pendulum is a pendulum consisting of a point mass at distance l from the pivot point. A physical pendulum is a rigid body that hangs from a pivot point.

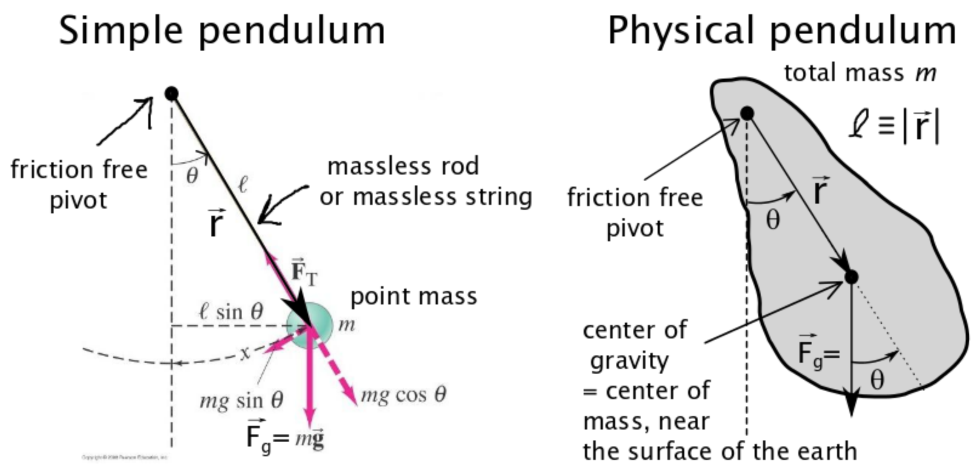

Figure 3.1: The schematics of the simple pendulum and the physical pendulum.

Assuming that the gravitational field is constant, e.g., near the surface of the Earth, the center of gravity[2] for the physical pendulum is then identical with its center of mass. We assume that the center of mass is at length l from the pivot point.

In both cases, the equation of motion in the rotational form is useful.

$$I\ddot{\theta} = \tau = -mgl\sin\theta \tag{3.10}$$

Here, we already used the fact that the torque[3] $\vec{\tau} = \vec{r} \times \vec{F}_g$ is given by, in both cases,[4] $\tau = -mgl\sin\theta$, since the angle between the vector $\vec{r}$, whose magnitude is l, and $\vec{F}_g$, whose magnitude is mg, is θ. And, the negative sign means that the direction of the torque is opposite to that of θ (positive as shown in the diagrams), meaning a **restoring torque**.

For small angle, $|\theta| \ll 1$ (Eq. 3.6), we get

$$\sin\theta \approx \theta \qquad |\theta| \ll 1 \tag{3.11}$$

using which the above Newton's equation of motion becomes

$$I\ddot{\theta} = -mgl\theta \tag{3.12}$$

[2]A physical pendulum in a non-constant gravitational field is complicated, since the center of gravity, at which the net force is exerted, will in general change as the pendulum swings.

[3]This is the net torque. In the simple pendulum case, the tension force $\vec{F}_T$ does not give rise to any torque on point mass m, since $\vec{F}_T \parallel \vec{r}$. In the physical pendulum case, any normal force that the physical pendulum experiences at the pivot is zero for the same reason.

[4]Note that here the symbol τ means the z component of $\vec{\tau}$, assuming the pendulum oscillation occurs in the xy plane.

This is, again, a SHO EOM, with

$$\omega = \sqrt{\frac{mgl}{I}} \qquad \text{physical (or simple; see below) pendulum} \tag{3.13}$$

It goes without saying that for the simple pendulum case, we can simplify this equation further, by using $I = ml^2$.

$$\omega = \sqrt{\frac{g}{l}} \qquad \text{physical pendulum} \tag{3.14}$$

Almost any pendulum around us is an example of a physical pendulum: a dining table lamp that is bumped by your head, or a big piece of meat hanging in a butcher shop.