Notes for Lecture 12

Lens and mirror equation

These notes emphasize things that you cannot find in the book, while not repeating what can be found in the book.

12.1 The master equation

For a spherical/plane optical element (mirror or thin lens), the following equation is valid

$$\frac{1}{f} = \frac{1}{d_o} + \frac{1}{d_f} \tag{12.1}$$

$$m = \frac{h_i}{h_o} = -\frac{d_i}{d_o} \qquad \text{lateral magnification} \tag{12.2}$$

under the usual assumption of small angles involved, or equivalently small object size compared to f.

It is worth noting that the derivation of the first equation ("the master equation" for thin lens or mirror) is based on the two simple facts: (i) a distant object is imaged at f and (ii) the light ray that hits the origin of the optical element passes through the optical element without any change (thin lens) or it is reflected symmetrically (spherical mirror). Using these two facts, and using a bit of trigonometry, the master equation can be proven both for a spherical mirror or a thin lens.

One main thing to note is that all of the quantities above can be negative. If d or f is negative, then the associated thing (object, image, or focus) is called "virtual." If d or f is positive, then the associated thing is called "real."

12.2 Compound element

One can combine many optical elements together to build a compound element—a telescope or a microscope, for example. In such cases, one must keep in mind that the image of one optical element becomes the object of the next optical element. What can be difficult at times is how to recognize the nature (real or virtual) object for an intermediate device. Here are some guidelines that must be heeded to carefully.

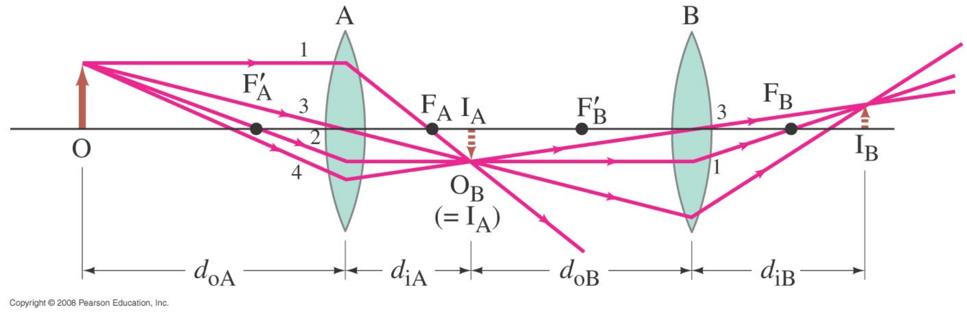
1. **Consider one optical element at a time!** When one draws a diagram for one optical element out of multiple elements used, the best practice is to forget temporarily all other optical elements and how the real light behaves! This is because the real-ness or the virtual-ness of object and iamge is determined only within the context of the diagram for *one* optical element of interest at a given time.

2. **When two optical elements are used in succession, then the image of the first optical element becomes the object of the second element.** This "chain rule" (already mentioned above) makes it possible to put together information that is gained by considering one optical element at a time. Note that just because the first element creates a virtual image does not mean that it is a virtual object for the second element! This real-ness and the virtual-ness of the same thing can change, as **the real-ness or the virtual-ness is determined per diagram of a given optical element** (see the previous item).

3. Applying Eq. 12.1 to one optical element at a time brings up the following important point: d and f are measured relative to the optical element in question. In other words, the zero of d or f is defined as the position of the optical element. So, in a multiple element problem, it is important to "shift the zero" of d and f properly as one shifts to a different optical element.

To appreciate what the above guideline really means, one must try to do actual problems. Here, we consider a few two lens systems to illustrate our points.

12.2.1 Example 1



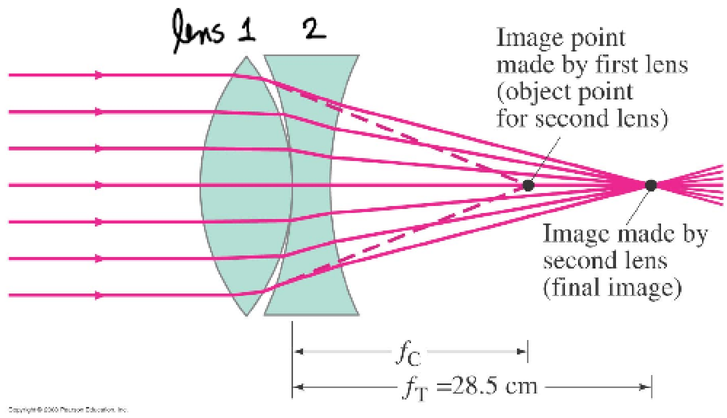
Here is an elementary example, a two converging lens system, with $d_{oA} > f_A$ and $d_{oB} > f_B$. First, the first lens (A) forms a real image (since $d_{oA} > f_A$) at d_{iA}. Second, this real image becomes a real object for the second lens (B), which then maps it to another real image at d_{iB}.

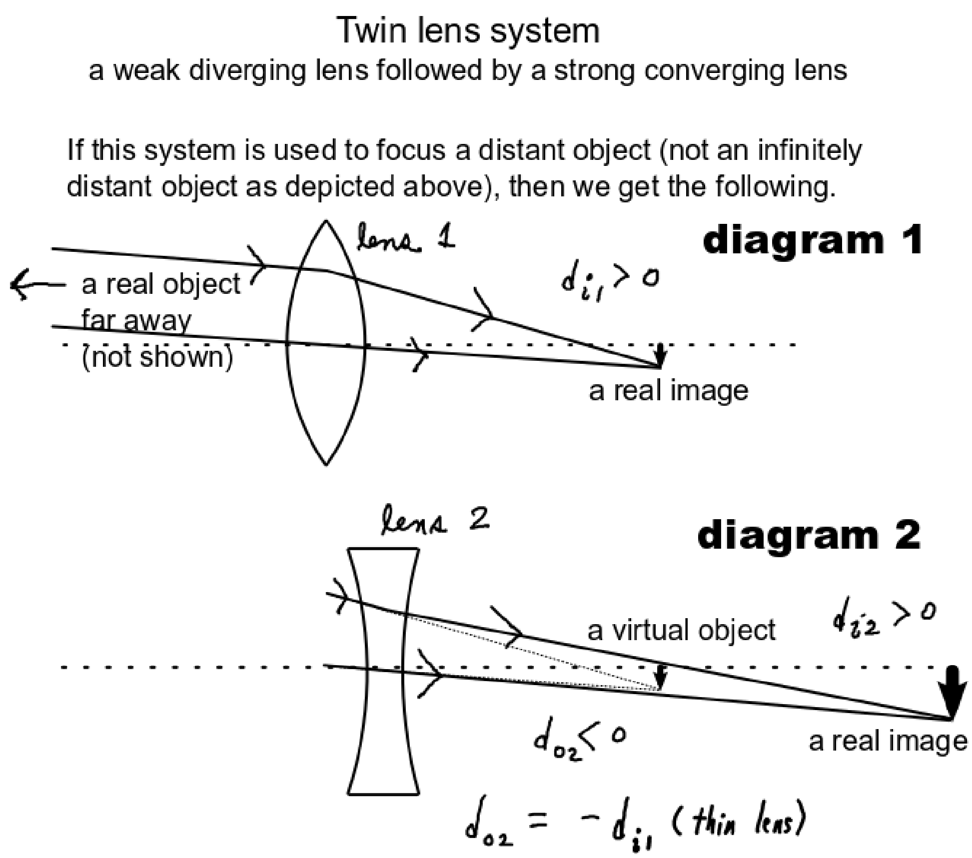
12.2.2 Example 2

Now, let us consider a slightly more complicated situation, as shown below. It is important to focus our attention to diagrams 1 and 2, as those diagrams, and only they, not the full diagram shown at the top, are relevant to Eq. 12.1, and to determining the real-ness or the virtual-ness of things.

Note that, *in diagram 1*, there *is* transmitted light around the image. So, the image is real. However, this image becomes the object for lens 2 (diagram 2). *In diagram 2*, there *is no* incoming beam of light around this object. So, we have a virtual object, and must use a negative value for d_{o2}. Assuming thin lenses, we get $d_{o2} = -d_{i1}$.

Just like in the mirror case, the above situation can be phrased in terms of the "right side" or the "wrong side." In all cases, what we refer to as the "right side" is where the relevant light (input light or output light) is. In the case of the mirror, the "right" side for the image is the front side, i.e., the same as the right side of the object, but, since a lens is a transmission device, the "right side" of the image for a lens is the opposite side to the right side of the object. With this in mind, note that in diagram 1, both the object and the image are on their respective right side, while in diagram 2, the image is, but the object is not.

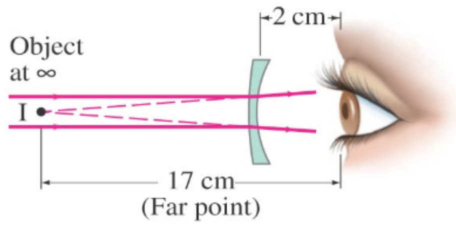




12.2.3 Example 3

Let us consider a third example involving a corrective lens of a near-sighted eye. Again, the situation is broken down into two diagrams. In diagram 1, note that there is no transmitted light around the image. So, we got a virtual image. We must use

$d_{i1} < 0$. However, this image acts as an object to the eye. From the eye digram (diagram 2), there *is* light around this object, and thus it acts as a real object. So, $d_{o2} > 0$.



This situation can be viewed as describing a two lens system, also. Lens 1 is the corrective lens shown. Lens 2 can be taken to be the lens of the eye (or more correctly, it must be taken as cornea + lens of the eye).

When this twin lens sytem is used to view a distant object (not at infinite distance), then the following diagrams apply to lens 1 and lens 2.

diagram 1 what lens 1 (corrective lens) sees

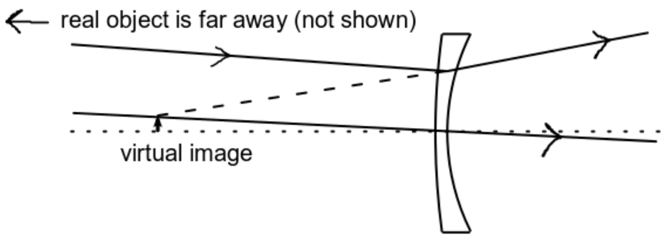
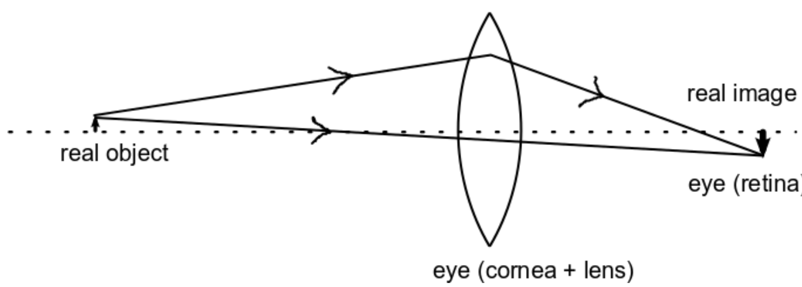
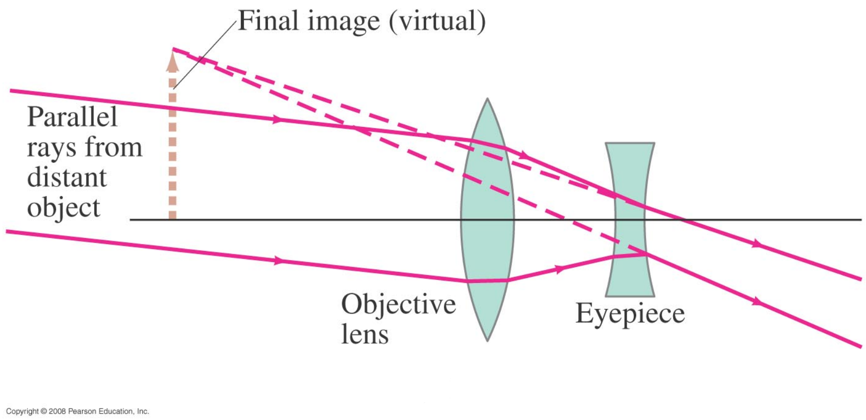


diagram 2 what lens 2 (eye) sees



12.2.4 Example 4

Lastly, let us consider the following situation, the design for a Galilean terrestrial telescope or an opera glass. Please answer these questions, yourself. Is the image of the objective lens real or virtual? Is the object of the eyepiece real or virtual? How about the image of the eyepiece?



12.2.5 Summary

To close, the following can be regarded as quick tricks, sometimes, for finding the realness and the virtual-ness of objects and images (including the focal length, which is d_i of an infinitely distant real object). Either 1 or 2 can be used.

1. If a thing is on the right side, then it is real. If on the wrong side, then it is virtual. The right side is defined as the same side where the relevant light is. The wrong side is defined as the opposite side to the right side. What divides these two sides? The optical element itself does so. So, an object is real if it exists on the side where the incoming beam is. An image that forms on the back of the mirror is virtual, since the relevant light, the reflected light, for the image is not there, but on the front side of the mirror. Virtual images of lenses exist on the wrong side where the incoming beam is, not on the right side where the transmitted beam is.

2. If the input beam is divergent on the optical element, then it means a real object. If the input beam is convergent on the optical element, then it means a virtual object. For an image, the opposite holds. If the output beam forms a converging beam, then the image is real. If the output beam forms a diverging beam, then the image is virtual.

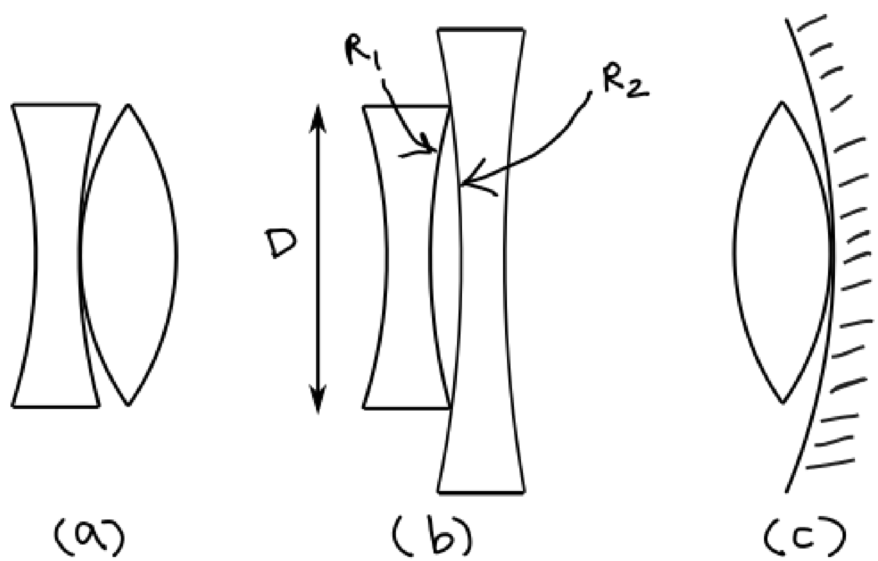
12.3 Thin compound element

Sometimes one stacks two or three optical elements without any gap to form a new compound optical element, and the compound element remains thin. In such a case, the following holds:

$$\frac{1}{f} = \sum_{i=1}^{n} \frac{1}{f_i}, \tag{12.3}$$

where f is the focal length of the total compound element, and f_i's are the focal lengths of individual optical elements.

Examples of this type of compound element include two thin lenses stacked together and a concave mirror on the ground with thin water on it. In the latter case, the number of individual elements (n) is three, since the light passes through the water lens twice. In the drawn examples below, $n = 2$ for (a) and (b), and 3 for (c), where a convex lens and a concave mirror are stacked.



One might ask, in a situation like (b), there has to be a gap between two lenses, even if you stack them up, so how can it be said that the compound element remains thin? You are asked to show that this is indeed true if $|R_1|, |R_2| \gg D$.

The proof of Eq. 12.3 is actually quite easy, if one notices that for all conceivable cases the real-ness or the virtual-ness of the image of the i-th element and the real-ness or the virtual-ness of the object of the i-th element, and that each optical element satisfies the master equation, Eq. 12.1. The key requirement is that the compound optical element still remains thin, so that the master equation, when applied to each element, can be thought to have the common origin, different from general cases, e.g., those considered in the last section except example 2. The actual proof of Eq. 12.3 is left for your work.

There is more! If f for the thin compound element is given as above, then we can continue to use Eq. 12.1 for the compound element as though it is by itself a thin lens or a spherical mirror. Why? As we noted above, the derivation of Eq. 12.1 depends on only two properties: the focal length and the thinness of the optical element. Subsequently, Eq. 12.2 can be used for a thin compound lens or mirror, also, since it is a bi-product of the same derivation.

More generally, note that $m = h_i/h_o$ is the definition of the lateral magnification, and so it remains valid for any optical device (thin or not) made of however many optical elements, but the same cannot be said of $m = -d_i/d_o$.