# Notes for Lecture 6

# Waves, Sound

We derived the wave equation in the last lecture note. As the wave equation derived is common to all wave phenomena that we deal with in this course, deriving it was good to do. In the last lecture note, we derived the wave equation for the sound wave in gas or liquid, but the same form of wave equation results for a string wave (lecture; book) or a longitudinal wave along a spring (slinky wave; also a model for the longitudinal sound wave in solid). Whichever system one considers, studying the derivation of the wave equation, one realizes that Newton's laws are the only underlying principles necessary for the derivation: in fact, the wave equation is just a way to re-state Newton's laws[1] that governs the wave motion.

Knowing the derivation of the wave equation is very nice. However, perhaps of more practical importance is to know how to appreciate and use the wave equation. In the last lecture note, we discussed the *the superposition principle*. The subtle difference between superposition (involving *virtual or real* source waves) and interference (involving *real* source waves) has been discussed as well. Here, in this lecture note, we continue to discuss the properties of wave, based on the wave equation whenever possible or convenient plus other principles such as the energy conservation.

## 6.1 Frequency

The box below summarizes a very useful fact. You will get a lot of mileage out of not forgetting this fact!

[1] On the other hand, the microscopic analysis that is necessary for the derivation of the wave equation can reveal nano-scale objects for which Newton's 2nd law is not valid, while Newton's 3rd law remains valid for them.

### The importance of frequency

When a wave propagates through various different media, through reflection, refraction, transmission, diffraction, etc., there is one thing that never[a] changes: frequency. Here, by *different media*, we mean not only different materials, but also the same material kept in different conditions, e.g., temperature, string tension (for string wave), pressure, and density.

[a]However, a wise student may note that the word "never" never really means never. Here in this course, we consider "linear systems" only, for which the frequency does not change. If a wave propagates in a non-linear medium, then the frequency can change. In general, any medium will act as a non-linear medium if the amplitude of the wave is large enough.

But, why is this fact true? It is because, when a wave propagates, what is happening is that a group of atoms shakes another group of atoms adjacent to them. Whether this "shaking" is happening between your hand and a string, between adjacent segments of a string, at the joint of two strings of different thickness, between air molecules and the molecules in your ear drum, or between water molecules and air molecules at the ocean surface, this shaking is coordinated, and the frequency is preserved.

We can also derive the unchanging nature of the frequency from the wave equation. In fact, many properties that we will discuss in the next few lectures are directly derivable from the wave equation itself, mathematically, although other general principles can be invoked for alternate derivations. This is not surprising, given that the wave equation is equivalent to Newton's laws, as mentioned in the introduction, and Newton's laws are consistent with other principles such as momentum conservation or energy conservation. The mathematical proof that the frequency must remain invariant is possible by noting *the boundary conditions* for the solution to the wave equation: $D$ must be continuous at all times when a medium changes abruptly (as we will discuss more in Section 6.8). Using this boundary condition, it is rather straightforward to show that the frequency must remain constant when the medium changes[2].

[2]Moreover, we can also prove other fundamental properties such as the law of refraction and the law of reflection using this boundary condition alone.

As noted in the footnote of the above box, all of these are true only for "linear media," which we are exclusively interested in in this course. Linear media are those systems for which effects going beyond Hooke's law can be ignored: in other words, the Hooke's law approximation (cf., Lecture 1) is very good for linear systems. When a non-linear medium is shaken at a certain frequency, its motion can involve additional frequencies different from the original frequency. Such non-linear effect is not captured in the wave equation that we derived in the last lecture—thus, we will gladly ignore any non-linear physics for the purpose of this course, as you will do even in many advanced courses of physics; but you will learn about non-linear physics in some later courses, and you will surely encounter it if you do any kind of real science or engineering projects.

Eventually, the best way to see why the frequency does not change is to see it through the energy conservation principle[3]. This is related to the fact that $\omega$ is deeply related to the energy of the wave, just as $k$ (wave number or wave *vector*) is deeply related to the momentum of the wave. However, proper appreciation of these two last sentences is possible only if you know some modern physics, i.e., quantum mechanics, and so they may be regarded as a mere passing remark at this point. However, when you do learn quantum mechanics later, you should re-think about all these classical waves that we are dealing with now in terms of quantum waves—I guarantee you that your eyes will widen with joy.

## 6.2 Polarization and speed

So far, we have been considering only a certain kind (transverse or longitudinal) of wave propagating through a given medium: for example, a string wave (transverse wave) or a sound wave in gas or liquid (longitudinal wave).

The name for this "kind" of wave is the **polarization**. Strictly, the polarization refers to the axis along which the local displacement $D(x,t)$ occurs. In our formulation so far, we have specified the $x$ axis as the axis along which the wave propagates. For this given axis of wave propagation, what are the possible polarization states? Note that for a string wave, then, there are two independent axes along which the string can be shaken ($y$ or $z$): so, there are two possible transverse polarization states. How about the sound wave in air (or water)? In this case, the local displacement of a slab of gas molecules or liquid molecules occurs along the same axis as the wave propagation. So, the polarization state for a longitudinal wave is one and only, as it

[3]Even in non-linear media, the energy conservation is valid. It is just that the consequence of the energy conservation is very simple (frequency remains the same) in linear media, but not simple in non-linear media.

coincides with the wave propagation axis by definition.

It is interesting to note that in some medium transverse polarization and longitudinal polarization can coexist. A seismic wave, or equivalently the sound wave in a solid, is such an example. A seismic wave propagating in a volume of earth is divided into a P (primary; pressure) wave or an S (secondary; shear) wave. The P wave is much like the sound wave in gas or liquid, except that the formula for the speed of wave is slightly different, involving Young's modulus $Y$ instead of bulk modulus $B$: $v = \sqrt{Y/\rho}$. The speed of the S wave is determined by the "shear modulus." The P wave is a longitudinal wave, and the S wave is a transverse wave, as we shall see in the next paragraph. Then, note that there can be two S waves, but only one P wave, if the direction of the wave propagation is given and the wave number is fixed, according to our discussion in the last paragraph.

How might one understand the pressure wave and the shear wave? Take a piece of solid and put it on a table. We can generate a sound on it and listen to it. The sound may be too weak to hear. In that case, take a screw driver, and make it touch the top of the solid, while the handle of the screw driver touches your ear. Now, have someone energetically tap the underside of the table with her or his palm. The sound that is generated will propagate to the solid and to your ear. Assuming that the solid is small, the sound propagates up the solid, while local displacements of thin layers of atoms in the solid also occur up and down. This is the P wave. Now, to generate an S wave, grind the solid on the table. This will also generate a sound wave, which propagates up the solid. However, the grinding action causes a horizontal movement of thin layers of atoms that make up the solid. This is the S wave. As you can see the S wave is a transverse wave, since the displacement is perpendicular to the wave propagation direction. Note that you can grind the solid in two independent directions on the table; so two S waves corresponding to two transverse polarizations.

It takes a bit of thinking to realize why S also stands for "secondary" in the seismic wave context. It is because the S wave propagates slower than the P wave. So, when an earth quake occurs, the P wave is the first wave to be detected, followed by the S wave. Why is the S wave slower? It is because the strain that is involved in a P wave is "harder" to generate than the "softer" strain that is required for an S wave. Why? Let us note that what we call "Hooke's law" eventually arises from tiny atoms and molecules that make up a solid. These molecules are compressed or elongated, when a P wave, or Young's modulus, is involved. On the other hand, they are slid sideways, when an S wave, or shear modulus, is involved. In both cases, the equilibrium is disturbed, and the system will exert a restoring force. However, in general, the same amount of displacement that is applied to compression or elongation gives rise to a greater amount of restoring force than if the same displacement is applied to sliding sideways. For one thing, note that the former will involve a local volume change, while the latter will not involve any local volume change. Young's modulus or shear modulus is in essence a measure of how large or small this restoring force is. For this reason, shear modulus tends to be roughly 2 to 3 times smaller than Young's modulus for the same material, causing the S wave to propagate slower.

## 6.3 Standing waves

A standing wave is as important as a travelling wave. In general, if a right moving wave $g(x - vt)$ (where $v > 0$) is superposed with a left moving wave $g(x + vt + x_0)$ of the same shape but with a possible offset in position, $x_0$, then the result is a standing wave: a wave that does not go anywhere, but whose (constant) energy makes the medium have a "breathing motion." So, one must keep in mind that for any standing wave the speed of the wave propagation is zero, while this is not true for a constituent wave, e.g., $g(x - vt)$ or $g(x + vt + x_0)$.

Just as studying the travelling sinusoidal wave is a fundamental way to start investigating any general travelling wave, studying the standing sinusoidal wave is of fundamental importance.

If we superpose the following two travelling sinusoidal waves ($A > 0$, as usual),

$$D_1(x,t) = A\sin(kx - \omega t + \phi_1), \tag{6.1}$$

$$D_2(x,t) = A\sin(kx + \omega t + \phi_2), \tag{6.2}$$

then we get a **standing sinusoidal wave**[4],

$$D_s(x,t) = D_1 + D_2 = 2A\sin(kx + \phi_a)\cos(\omega t + \phi_d), \tag{6.3}$$

where new phase constants are defined as

$$\phi_a = \frac{\phi_1 + \phi_2}{2}, \tag{6.4}$$

$$\phi_d = \frac{\phi_2 - \phi_1}{2}. \tag{6.5}$$

Note that the above form of standing wave can be understood as having a *time-independent* shape, $\sin(kx + \phi_a)$, which is multiplied by the *time-dependent* overall

[4]Here, we are using a trigonometric identity, $\sin a + \sin b = 2\sin\left(\frac{a+b}{2}\right)\cos\left(\frac{a-b}{2}\right)$.

scaling factor, given by $2A\cos(\omega t + \phi_d)$. So at any given point of time, the wave form looks like $\sin(kx + \phi_a)$ or $-\sin(kx + \phi_a)$ depending on what the sign of the overall scaling factor $2A\cos(\omega t + \phi_d)$ is. Furthermore, for special values of $t$ for which $\omega t + \phi_d = \frac{\pi}{2} + n\pi$ ($n$ = integer), we get $D_s(x,t) = 0$, temporarily. The result: we get a sine curve that "breathes" due to the overall scaling factor changing from $2A$ to $-2A$ and then back to $2A$ in one period.

This breathing motion of a standing wave is quite different from a travelling wave (e.g., $A\sin(kx - \omega t + \phi)$), which has a *fixed* shape that travels as a function of time.

However, observe that, for both a travelling sinusoidal wave and a standing sinusoidal wave, the local motion at a fixed $x$ value is a simple harmonic motion (SHM), due to the wave function ($D$) having a sinusoidal function in time. A difference is this: in a travelling sinusoidal wave, the amplitude is independent of $x$, while in a standing sinusoidal wave, the amplitude is dependent on $x$. In the above example, the amplitude of the local SHM at $x$ is given by $2A|\sin(kx + \phi_a)|$.

## 6.4 Energy, power, and intensity

From the theory on work and energy, you learned that power is defined as the energy delivered per unit time. The **intensity** is the power delivered per unit area.

A quantity that is very closely related to intensity is the so-called **flux**. As the name suggests, the flux describes a flow, and is a vector quantity. The intensity is the magnitude of the energy flux, the flow of energy per unit time and unit area.

Let us suppose that wave energy, $E$, travels distance $L$ in time $T$ passing perpendicular to the surface whose area is $S$. Thus, the product $LS$ gives the total volume swept by this energy flow. What is the area $S$? For a string wave, this can be thought of as the cross sectional area of the string. For a sound wave in a large volume of gas or liquid, it can be taken to be any appropriate area of your choice: for instance, you might like to examine a circular area—then the volume represented by $LS$ will correspond to a cylinder–or a square area—then the volume represented by $LS$ will correspond to a rectangular cylinder. In any case, the following holds.

$$\begin{aligned} P &= \frac{E}{T}, && P = \text{Power, } E = \text{Energy, } T = \text{Time} \\ I &= \frac{P}{S} && I = \text{Intensity, } S = \text{Surface area} \\ &= \frac{E}{TS} = \frac{E}{LS}\frac{L}{T} = \rho_E v. \end{aligned}$$

Here, $\rho_E$ is energy per volume, i.e., the energy density, and $v$ is the speed of the wave propagation.

What we just did is valid only when the energy flow is uniform. A more generally correct way is to consider infinitesimal quantities: a small energy $dE$ transported by wave through an area $S$ over a small distance $dx$ (perpendicular to the surface corresponding to $S$) in a small time $dt$. If the energy flow is not uniform in space, then $S$ can be taken to be a small area so that the energy flow through that small area $S$ is uniform. Then,

$$P = \frac{dE}{dt},$$

$$I = \frac{P}{S} = \frac{dE}{Sdt} = \frac{dE}{Sdx}\frac{dx}{dt} = \frac{dE}{dV}\frac{dx}{dt} = \rho_E v. \qquad \rho_E \equiv \frac{dE}{dV}$$

Here, $dV$ is the infinitesimally small volume $dV = Sdx$. What did we gain by considering infinitesimals? We have shown that $I = \rho_E v$ is valid even if $\rho_E$ (or $v$!) is explicitly dependent on time and space so that $I$ is explicitly dependent on time and space. This is great! However, no worries, if this sounds a bit too complicated—in this course, we will focus on simple cases where $I$ is constant, and so either formalism suffices.

That the intensity is given by $\rho_E v$ is no coincidence, and is worth remembering, since the intensity is the magnitude of the energy flux. For many physical quantities (like charge, energy, number) its flux (i.e., charge flux = current density, energy flux, number flux) is given by the density of that quantity (i.e., charge density, energy density, number density) times the velocity at which that quantity (i.e., charge wave, energy wave, number wave) flows.

To summarize:

$$I = \rho_E v. \qquad \text{Intensity. } \rho_E = \frac{dE}{dV} = \text{energy density.} \tag{6.6}$$

One word of caution: the expression $v$ in $I = \rho_E v$ is *not* necessarily equal to $v$ in the wave equation. Recall that a standing wave does not propagate. For a standing wave, the energy simply does not flow; instead, it just exists. Nevertheless, $D$ for a standing wave satisfies the wave equation with a finite $v$, which should not be interpreted as the wave speed for a standing wave: in this case $v$ is simply the speed of one of the two counter-propagating waves that superpose to make the standing wave.

Now, *what* is the energy stored in a wave? A very general way to answer this question is to say that it is the sum, i.e., the integral, of the infinitesimal energies, each of which ($dE$) corresponds to the energy of the mass element $dm$ between $x$ and $x + dx$ in equilibrium. Thus, the total energy in a wave is $\int_0^L dE$, supposing that we are considering a medium of length $L$. For a general wave, this integral is not a trivial one, since each mass element goes through a complicated motion, giving a complicated expression for $dE$. The good news is that we can always decompose any wave into a Fourier sum/integral of sinusoidal waves, for each of which $dE$ is very simple to compute. Then, it can be shown that the *average* energy of the original wave is the sum of all the energies calculated for constituent sinusoidal waves.

So, we consider a sinusoidal wave only in this section. For any sinusoidal wave, computing $dE$ is very easy, since each mass element goes through a sinusoidal motion, i.e., a simple harmonic motion (SHM). In particular, **we will focus on a travelling sinusoidal wave (TSW) for the rest of this section**, leaving the consideration of the energy of a standing sinusoidal wave to readers.

Consider the mass element $dm$ of the medium between $x$ and $x+dx$ in equilibrium. The energy, $dE$, that $dm$ acquires if it goes through a SHM with amplitude $A$ and angular frequency $\omega$ is given by

$$dE = \frac{1}{2}\, dm\, A^2\omega^2. \tag{6.7}$$

Note that this expression is the total energy for $dm$, since it is just the maximum kinetic energy expression for mass $dm$ undergoing a SHM. Now, imagine that the medium has a uniform linear mass density, $\mu$, as we typically assume. Then, $\mu = dm/dx$, and thus we get

$$dE = \frac{1}{2}\, \mu dx\, A^2\omega^2. \qquad \text{Energy (infinitesimal, TSW)} \tag{6.8}$$

Integrating this over any length $L$ of the medium is easy for a travelling sinusoidal wave, since the amplitude of the local SHM is $A$, independent of $x$. That is, the right hand side of the above equation is just a constant times $dx$, and so we get

$$E = \frac{1}{2}\, \mu L\, A^2\omega^2. \qquad \text{Energy (TSW) for medium of length } L \tag{6.9}$$

Now, how about the power? Either by doing $dE/dt$ or $E/T$ and noting either $dx/dt = v$ or $L/T = v$ respectively, we get

$$P = \frac{1}{2}\mu v A^2 \omega^2. \qquad \text{Power (TSW)} \qquad (6.10)$$

Dividing this by the surface area $S$, and noting that the volume density $\rho = \mu/S$, we get the intensity

$$I = \frac{1}{2}\rho v A^2 \omega^2 \qquad \text{Intensity (TSW)} \qquad (6.11)$$

$$= 2\pi^2 \rho v A^2 f^2. \qquad \because \quad \omega = 2\pi f \qquad (6.12)$$

The above expressions for $dE$ or $E$ can be rewritten as $dE = \frac{1}{2}\rho A^2 \omega^2 S dx$ or $E = \frac{1}{2}\rho A^2 \omega^2 S L$, respectively, using $\mu = \rho S$. Noting that $Sdx = dV$ or $SL = V$ ($V$ = volume), we see that $I = \rho_E v$ where $\rho_E = dE/dV = E/V$ is the energy density, as we already anticipated.

## 6.5 Wavefront, spherical wave

A travelling sinusoidal wave, $D(x,t) = A\sin(kx - \omega t + \phi)$, is often referred to as a sinusoidal **plane wave**. If the spatial dimension is one, then a travelling sinusoidal wave is all we need to consider, and calling it a plane wave is a moot point. However, we live in the three dimensional world. Because of this, the more proper way to write the wave displacement is $D(x,y,z,t)$, not $D(x,t)$. Then, a sinusoidal wave travelling along the $x$ direction can be written as

$$D(x,y,z,t) = A\sin(kx - \omega t + \phi). \qquad \text{sinusoidal plane wave} \qquad (6.13)$$

Indeed, one realizes that this is what we *really meant* all along for wave propagating in a three dimensional medium (like sound wave propagating in air), omitting $y$ and $z$ for simplicity.

Let us consider the **wavefront**, which is defined by the collection of points sharing a common phase. In the above equation, the phase is $kx - \omega t + \phi$, and so setting this to be a constant (say $\phi_0$), we get

$$x = vt + \frac{\phi_0 - \phi}{k}. \qquad \text{wavefront for sinusoidal plane wave;} \quad v = \omega/k \tag{6.14}$$

This wavefront equation will describe the wavefront of the crest if $\phi_0 = \pi/2$ (modulo $2\pi$), the node if $\phi_0 = 0$ (modulo $\pi$), the trough if $\phi_0 = 3\pi/2$ (modulo $2\pi$), etc. Since $x$ = constant is the equation of a plane (perpendicular to the $x$ axis), the wave front in this case is a plane at any fixed point of time, justifying the name "plane wave." In addition, the wave front moves at velocity $v = \omega/k$ along the $x$ direction, which is *why* we can interpret $\omega/k$ as the speed of a sinusoidal plane wave, as we have been doing for some time now.

Let us now consider a point source of waves. A person singing in a room, a light bulb lit in a room, or the Sun can be thought of as a point source of wave (sound or light) to a good approximation. How should we represent the wave that emanates from such a source? It is clear that a plane wave will not do, since the wave from such a point source propagates in all directions, rather than along a particular axis.

The proper wave to consider for these cases is the **spherical wave**:

$$D(x,y,z,t) = \frac{A}{r}\sin(kr - \omega t + \phi). \qquad \text{sinusoidal spherical wave; } r \equiv \sqrt{x^2+y^2+z^2} \tag{6.15}$$

Here, $A > 0$ is a constant, just as in a plane wave, but its dimension is now different—it is length squared, instead of length. The wavefront for a spherical wave is a spherical surface, described by the following equation, obtained by equating the phase $kr - \omega t + \phi$ to a constant $\phi_0$.

$$r = vt + \frac{\phi_0 - \phi}{k}. \qquad \text{wavefront for spherical wave}, v = \omega/k \tag{6.16}$$

So, the wave propagates as a spherical wavefront moving at speed $v = \omega/k$. Now, let us take it upon ourselves to prove why we can take the amplitude of the sinusoidal spherical wave to be $A/r$. Namely, writing the sinusoidal spherical wave as $D(x,y,z,t) = A_s(r)\sin(kr - \omega t + \phi)$, why is it that $A_s(r)$ is given by a constant divided by $r$? A physical explanation is given in the next few paragraphs.

First, it is important to note that even a spherical wave will look like a plane wave when examined in a small space. This is not surprising since we live on a round planet, but we recognize our local landscape as being very flat. So, consider the spherical surface at radius $r$ from the origin, where the source of the spherical wave lies. We can divide up the spherical surface into very small patches, each of which can be considered as perfectly flat to a good approximation and having small surface area $dS$. When we sum up all these small surface areas, we must get the total surface area of the sphere, $4\pi r^2$. That is,

$$\int_{\text{spherical surface at } r} dS \;=\; 4\pi r^2. \tag{6.17}$$

Let us choose an arbitrary one of these small patches of area $dS$. We can define, without loss of generality, the line connecting the origin to this chosen patch as the $x$ axis. The spherical wave that passes this particular patch is approximated very well by the formula $D(x,y,z,t) \approx A_s(r)\sin(kx - \omega t + \phi)$, since $r \approx x$ for points on this patch, since the patch is infinitesimally small by construction. Also, for this particular patch, $r$ (or $x$) is constant, and so $A_s(r)$ is a constant, if examined on this patch; therefore, we just *proved* that a spherical wave looks like a plane wave, if examined locally in space.

Second, we consider the energy passing per unit time through this particular patch of surface area $dS$. This is easy to do since we now know all about energy, power, and intensity for any sinusoidal plane wave. For instance, Eq. 6.11 or 6.12 will do for the intensity. Using Eq. 6.12, we can write $I = CA_s^2$, where $C = 2\pi^2\rho v f^2$. Note that $C$ is independent of $r$, while $A_s$ may be dependent on $r$, as we are trying to find out.

Third, we now consider **the principle of energy conservation**: the energy that propagates through the total spherical surface at radius $r$ per unit time must be independent of $r$. That is, the power delivered to the total spherical surface is independent of $r$. This statement indeed represents the principle of the energy conservation, if one recognizes that the wave speed is independent of $r$ (or any other positional parameter) in a homogeneous medium, as assumed. The power that passes through a patch of area $dS$ is given by $dP = IdS$. Since $I = CA_s^2$ is a constant over the spherical surface (i.e., it is independent of the direction), we can easily integrate $dP$ over the entire spherical surface to get the total power $P$:

$$P \;=\; \int_{s.s.r} dP \;=\; \int_{s.s.r} I dS \;=\; I \int_{s.s.r} dS \;=\; I\, 4\pi r^2, \tag{6.18}$$

where $s.s.r$ means the spherical surface at $r$ and Eq. 6.17 was used in the last step. Now, using $I = CA_s(r)^2$, we get

$$P \;=\; 4\pi C\, A_s(r)^2 r^2 \;=\; \text{independent of } r \text{ (energy conservation).} \tag{6.19}$$

Therefore, we arrive at the results that we set out to prove:

$$A_s(r) \propto \frac{1}{r}, \qquad D = A_s \sin(kr - \omega t + \phi) \tag{6.20}$$
$$I \propto \frac{1}{r^2}. \qquad \text{the “inverse square law”} \tag{6.21}$$

That is, we just proved Eq. 6.15, and the inverse square law for spherical waves.

## 6.6 The nature of the SHM participating in a wave

When we considered the energy of a sinusoidal wave, we found it convenient to divide by the medium into small pieces of equal length, $dx$, in equilibrium, and then consider the energy contained in a small mass element $dm = \mu dx$ during the wavy motion. For either travelling or standing sinusoidal wave, the motion of such a mass element is a SHM. Let us analyze this situation in a more microscopic way.

Recall the wave equation

$$\frac{\partial^2 D}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 D}{\partial t^2}. \qquad D = D(x,t),\ v > 0. \tag{5.1}$$

For either standing or travelling sinusoidal wave, $D(x,t) = a\sin(kx + b)$ where $a$ and $b$ are $x$-independent. Therefore, the left hand side gives $-k^2 D$. Thus, the wave equation becomes

$$-k^2 D = \frac{1}{v^2}\frac{\partial^2 D}{\partial t^2} \tag{6.22}$$
$$\therefore \quad \ddot{D} = -(kv)^2 D. \qquad \text{sinusoidal wave, standing or travelling} \tag{6.23}$$

As expected, this is a SHO EOM for the displacement $D$ of the particle, which is the mass element $dm$. Note that $x$ is not a dynamical variable, since it is merely an index to identify different particles, i.e., different mass elements. $D$ is the dynamical variable that shows a SHM. The angular frequency can be read off from the above EOM as $\omega = kv$, since the SHO EOM for $D$ must read $\ddot{D} = -\omega^2 D$.

The fact that $D$ is going through a SHM means that there is an effective Hooke's law force acting on $D$ given by $-\kappa D$, where we use $\kappa$ for spring constant instead of $k$, which is already taken to represent wave number. Given mass $(dm)$ and angular frequency $(\omega)$, we can determine the spring constant as $\kappa = dm\,\omega^2$ (Eq. 1.9).

Here is a question that one can ask. **Is $\kappa$ an inherent natural property of the medium? The answer is no! It is rather a property determined by the external driving frequency $\omega$.** Here is what I mean—for a fixed $dx$ value and a fixed $\mu$ value, $\kappa$ is entirely determined by $\omega$, which is determined by the source of the wave! So, it would be a mistake to think that $\kappa$ is somehow fixed by a natural property of the particle in question and its interaction with neighbors. This is different from the SHO problem that we discussed in Lectures 1 through 3, where the Hooke's law existed prior to any consideration. In general, in a sinusoidal wave phenomenon, all there is is a *synchronized* set of SHO motions of all particles.

Note that in some cases the medium in which a sinusoidal wave is generated is itself a spring. For instance, a wave can be generated in a slinky or in a solid, which may be crudely modeled as a bunch of interlocked springs. From the above discussion, it should be clear that $\kappa$ is not equal to the spring constant of the slinky or the solid, or any part of it. $\kappa$ depends on $\omega$, but the spring constant of a slinky is a constant. In other cases, the medium in which the wave propagates does not involve any local spring: string wave or sound wave in air is a good example.

Nevertheless, regardless of the medium, when a sinusoidal wave is generated, each small mass element goes through a SHM. This is not because there is an inherent Hooke's law force on each mass element that gives rise to such a SHM but because the forces that act on each mass element (some of which may be real spring forces) sum up to a force that looks just like a Hooke's law force, responsible for such a SHM.

## 6.7 Natural frequency, resonance

What we discussed in the last section can be summarized as: you shake a wave medium at a certain frequency, then the medium will just copy that frequency, giving the appearance of a SHM for each small part of the medium.

This summary is true and useful, but it does paint a picture of a dull featureless passive medium. Do not be disappointed. The reality of science is never really dull.

Even a string is not really dull. You can simply tie the string at both ends, and voila—you have a musical instrument (a guitar string)! Even a volume of air is not dull. You can trap it in a bottle or a tube, and you have a musical instrument with a characteristic note!

What happens is that a guitar string or a trapped air in a tube develops a **natural frequency** or a **resonance frequency**. We shall derive what these resonance frequencies are in future lectures. For now, it suffices to know that these natural frequencies ($\omega_n$) are discrete, e.g., $\lambda_n = 2L/n$ and $\omega_n = vk = 2\pi v/\lambda_n$ for a guitar string with $n = 1, 2, 3, ...$

When a wave medium has a natural frequency (or a set of natural frequencies), it responds with much greater sensitivity at that frequency. This is why even a bridge can show a very wild oscillation, caused by a blowing wind, to the point of collapse. This is also why when you blow into a wind instrument or pluck a guitar string in somewhat arbitrary manner, creating a continuous distribution of frequencies initially, the sound that comes out from the instrument is a reproducible tone—the natural frequency of the instrument—since that is where the instrument responds with the greatest sensitivity. For sure, any sinusoidal motion that you may force on a medium will be replicated by the medium, but if the frequency of the sinusoidal motion is just the natural frequency of the medium, the medium will be able to accommodate that motion most easily.

We can rephrase the above observation regarding natural frequencies of a musical instrument as follows. When you excite a musical instrument with a wave pattern that is a superposition of waves with various frequencies, pretty soon only those wave components corresponding to resonant frequencies will survive.

The reason why the system reacts most easily at a resonance frequency is because the system can sustain the motion corresponding to the resonance frequency *even without any external driving force* if it happens to have the required energy for the motion and if there is no damping mechanism.

Here is an example. Consider a guitar string of length $L$. Its resonant frequencies are given by $\omega_n = 2\pi v/\lambda_n = n\pi v/L$, where $v = \sqrt{F_T/\mu}$ and $n = 1, 2, 3, ...$ If the guitar string happens to oscillate at one of these frequencies, then it will be able to sustain that mode for a long time. However, if the guitar string happens to oscillate at a frequency different from any resonant frequencies, then it is not possible to do so. The reason is simple—the natural frequencies involve the motion of the guitar string that satisfies the boundary condition, i.e. the amplitude $D$ must vanish at the end points. So, as long as there is no damping, the motion will be self-sustained forever, once started. Even with damping, the resonant mode lasts long. Waves generated at non-resonant frequencies will self-annihilate quickly due to destructive interference and damping.

## 6.8 Reflection, transmission, and phase shift

Let us consider a plane wave impinging on the boundary of medium 1 and medium 2. We take the coordinate system such that $x = 0$ defines the plane of boundary between the two media and medium 1 exists in the $x < 0$ region while medium 2 exists in the $x > 0$ region. Let us suppose, in addition, that we send an initial wave, a sinusoidal plane wave, from left to right, starting from $x = -\infty$. Some of this wave will be reflected and some transmitted. Then, the displacement field, $D$, for the wave can be written as[5]

$$D(x,t) = \begin{cases} A\sin(k_1 x - \omega t + \phi) + A_R \sin(-k_1 x - \omega t + \phi), & x \le 0 \text{ (medium 1)} \\ A_T \sin(k_2 x - \omega t + \phi). & x \ge 0 \text{ (medium 2)} \end{cases} \tag{6.24}$$

The first observation to be made is that, even when the medium changes abruptly, $D(x,t)$ must remain continuous. This makes physical sense as long as we are considering the usual case when the overall medium remains continuous. More mathematically, the continuity of $D(x,t)$ can be concluded by examining the wave equation near the boundary[6].

The continuity of $D$ at origin means that $A\sin(-\omega t + \phi) + A_R \sin(-\omega t + \phi) = A_T \sin(-\omega t + \phi)$. This leads to

$$A + A_R = A_T. \tag{6.25}$$

The second observation to be made is that $\frac{\partial D}{\partial x}$ may or may not be continuous at the boundary. This depends on different waves (continuous for string wave, but discontinuous for sound wave). One might ask, then, trying to solve for the two unknowns $A_R$ and $A_T$ in the above equations, how might one approach the problem, in general? The answer is to use the **energy conservation principle**, instead, which is a way to obtain the second equation, equivalent to considering the boundary condition for $\frac{\partial D}{\partial x}$.

Recall from Eq. 6.11 that $I \propto \rho v A^2$, where the factor $\frac{1}{2}\omega^2$, which is independent of medium, is left out. By energy conservation, the power that goes into the boundary region must be exactly equal to the power that comes out of it, in a steady state. In the current case where the waves are directed perpendicular to the boundary surface,

[5]Compared to that textbook treatment (problem T15-40), our $A_R$ has an overall sign change, due to the fact that the reflected part of the wave is written differently here. It is more physically transparent to write the reflected wave, as we do here, i.e., as $A_R \sin(-k_1 x - \omega t + \phi)$, since what happens to a reflected wave for the phase factor, $kx - \omega t + \phi$, is the reversal of the direction compared to the original wave, i.e., the reversal of the wave *vector* ($k \to -k$), which is the true fundamental nature of what we have been calling the wave number (cf., LN 4).

[6]This step requires extra caution, and it is a somewhat advanced topic that we will not go into here. However, the following general comments might be helpful. The wave equation as we know it, $\frac{\partial^2 D}{\partial x^2} = \frac{1}{v}\frac{\partial^2 D}{\partial t^2}$, is valid only for a uniform medium. In the region where the medium changes, this wave equation must be modified (e.g., for sound wave) or remain valid (e.g., for string wave). If the modification is necessary, then the modified wave equation must be examined.

this means that the intensity that goes into the boundary region is the same as the total intensity that comes out of it[7] In other words,

$$\rho_1 v_1 A^2 = \rho_1 v_1 A_R^2 + \rho_2 v_2 A_T^2. \tag{6.26}$$

For string wave, we get $\rho_i v_i \propto \sqrt{\rho_i} F_T \propto \frac{1}{v_i} \propto k_i$ $(i = 1, 2)$, where again the factors that are independent of medium are left out. It turns out that $\rho_i v_i \propto k_i$ for light as well. For sound wave (gas or liquid), we get $\rho_i v_i = \sqrt{B_i \rho_i}$ where $B_i$ is the bulk modulus in medium $i = 1, 2$. Therefore, the above equation can be written as

$$a_1 A^2 = a_1 A_R^2 + a_2 A_T^2. \tag{6.27}$$

$$a_1 = \rho_1 v_1, \quad a_2 = \rho_2 v_2. \tag{6.28}$$

$$a_1 = k_1, \quad a_2 = k_2. \quad \text{string wave, light; note that } k_i \propto 1/v_i \tag{6.29}$$

$$a_1 = \sqrt{B_1 \rho_1}, \quad a_2 = \sqrt{B_2 \rho_2}. \quad \text{sound wave (gas, liquid)} \tag{6.30}$$

$$a_1 = \sqrt{Y_1 \rho_1}, \quad a_2 = \sqrt{Y_2 \rho_2}. \quad \text{sound wave (longitudinal sound in solid)} \tag{6.31}$$

Clearly, a mixed case variant of the two last cases is possible; if a solid medium meets a gas or liquid medium, one can use $Y$ for the solid side and $B$ for the gas or liquid side.

We have two equations[8], Eqs. 6.25 and 6.27, which can be solved to give, $A_R$ and $A_T$ in terms of $a_1$, $a_2$, and $A$.

$$A_R = \frac{a_1 - a_2}{a_1 + a_2} \cdot A, \tag{6.32}$$

$$A_T = \frac{2a_1}{a_1 + a_2} \cdot A. \tag{6.33}$$

These are important results. To appreciate these results, let us assume $A > 0$, as we always do. Since $a_i > 0$, we see that $A_T > 0$, always. However, the sign of $A_R$ is given by the sign of $a_1 - a_2$.

[7] Here, the assumption is that the area through which the wave propagates is constant throughout different media. This may not be the case, if two strings with different thicknesses are connected, or sound or light is refracted at an oblique angle, not at normal angle. In these cases, $a_1$ and $a_2$ must be modified. However, for the case of two strings with different thicknesses, the results $a_1 = k_1$ and $a_2 = k_2$ still remain valid.

[8] For the sound wave, the energy conservation equation can be replaced by the continuity equation for $\Delta P = -B\frac{\partial D}{\partial x}$, which is physically plausible and leads to the same result. For the string wave or the light wave, as treated here, the energy conservation equation can be replaced by the continuity equation for $\frac{\partial D}{\partial x}$, which can be justified from the wave equation, and gives the same result.

For string wave or light wave $a_i = k_i \propto 1/v_i$, and so if the speed of wave is slower in medium 2, then $a_2$ is greater than $a_1$ and $A_R < 0$. For sound wave, the opposite is true, in general. Usually, $\rho$ and $v$ go together, or $B$ and $\rho$ go together. And so, the medium with a higher value of $v$ will have a higher value of $a$ as well. In any case, the negative sign for $A_R$ occurs, if sound wave is reflected off of a denser and higher speed medium, e.g., when sound wave propagating in air is reflected off of water or solid.

Here is a summary.

- $A_T > 0$, always.
- $A_R > 0$ if $a_1 > a_2$ ("no phase shift").
- $A_R < 0$ if $a_1 < a_2$ ("$\pi$ phase shift").

The reason why we say that there is a $\pi$ phase shift in the last case is the following. If $A_R < 0$, then the reflected wave, $A_R \sin(-k_1 x - \omega t + \phi)$ does not conform to our definition of a travelling sinusoidal wave, as defined in Eq. 4.1, since $A_R < 0$. However, we can make this functional form conform to Eq. 4.1, if we note

$$A_R \sin(-k_1 x - \omega t + \phi) \;=\; -A_R \sin(-k_1 x - \omega t + \phi + \pi)$$

since $\sin(\star + \pi) = -\sin(\star)$. Now, the right hand side is in the standard form where the amplitude $(-A_R)$ is positive. This is made possible by adding $\pi$ (module $2\pi$) to the phase, and so the "$\pi$ phase shift."

Let us consider the string wave case. We will leave it to readers to think about situations for sound or light, analogous to what we will consider for strings waves in this paragraph and the next paragraph. Suppose a string with a certain density (medium 1) is connected to another string with a different density (medium 2). If medium 2 is a string with a higher density, then $v_2 < v_1$, since $v = \sqrt{F_T/\mu}$ and $F_T$ must be the same. And so, $a_1 < a_2$, since $a_i \propto 1/v_i$ for string wave. So, when the wave is reflected from medium 2, the wave is phase shifted by $\pi$. However, if medium 1 has a higher density than medium 2, then there is no phase shift.

Other cases of interest are when a string has a fixed end, or a free end. If a string is fixed at $x = 0$, then it means that $D(x = 0, t) = 0$. This can occur only if $A_R = -A$. That is, the $\pi$ phase shift of the reflected wave is a must for there to be a fixed end, and additionally $|A_R| = A$ is required, which makes sense in terms of the energy conservation. If a string has a free end at $x = 0$, then it means that $A_R = A$. These two cases can also be modeled using the above setup if we take $a_1 \ll a_2$ (fixed end; medium 2 has an infinite density) or $a_2 \ll a_1$ (free end; medium 2 has a vanishingly small density): indeed these conditions lead to $A_R = -A$ or $A_R = A$ respectively.

An interesting way to think about the phase shift physics discussed in the previous two paragraphs is to think about the propagation of a square pulse wave. At the edge of such a square pulse, the physics is a one dimensional elastic collision between two mass elements. In particular, at the boundary between two different strings, the physics is that of the one dimensional collision between two different masses. The physics of phase shifts discussed in the above two paragraphs can then be explained succinctly in terms of the collision physics. That is, when a heavy particle, initially moving, pulls a light particle, initially at rest, then both particles move in the same direction (as the initial velocity of the heavy particle) in the final state. On the other hand, when a light particle, initially moving, pulls a heavy particle, initially at rest, then the light particle changes its direction of motion, while the heavy particle moves in the same direction as the initial velocity of the light particle.