Notes for Lecture 5

Waves, Sound

5.1 Wave equation

For many, but not all, waves, the following wave equation holds. For instance, it holds for string wave, sound wave, and light.

$$\frac{\partial^2 D}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 D}{\partial t^2} \qquad D = D(x,t),\ v > 0 \tag{5.1}$$

For the purpose of this course, we can call this *the* wave equation. But, you can keep in the back of your mind that there might be other wave equations in physics—you will encounter them in advanced courses. One thing that you can already know is that many qualitative features of wave are shared by different waves satisfying different wave equations—the most important property is perhaps the superposition principle, which we will discuss at the end of this LN.

The above wave equation describes a wave propagating at speed v. The above equation is valid even when v is not a constant, in the sense that it depends on k. Since $\omega = vk$ for a travelling sinusoidal wave, we see that when v is dependent on k, $\omega(k)$ is a non-linear function of k. In such a case, the medium is said to be **dispersive**. Generally, the function $\omega(k)$ is defined as the **dispersion relation**, whether or not v is k-dependent.

5.2 Wave equation for string wave

One can ask why is the string wave speed given by $\sqrt{F_T/\mu}$? To derive this, one can examine, for example, the front end of a pulse, and apply Newton's law, as was done in class (cf., Section T15.2). Or, one can examine the general Newton's law equation of motion for a random part of the wave, and derive the wave equation, as was done in class also (cf., Section T15.5). Using the latter approach is more satisfactory, since using it we can derive the wave equation, Eq. 5.1, itself.

5.3 Wave equation for longitudinal sound

Here, we derive the same wave equation for a longitudinal sound wave. We shall see that the same equation results, where $v = \sqrt{B/\rho}$. By applying the same derivation, it is also straightforward to verify that $v = \sqrt{Y/\rho}$. Both these expressions had been given, and explained, near the end of the last LN.

Note to readers: On first reading, you can skip most of the following (longish) derivation, note only those things noted in boxes, and jump to the next section where the *solutions* to the wave equation are discussed. The derivation here may be just a bit trickier than the string wave case, because $D(x,t)$ and x are along the same direction for a longitudinal wave.

It seems useful to derive the wave equation for this particular problem, since it illustrates many points. While every student should follow this derivation, it is not required that she/he knows to how to reproduce the derivation. However, it is crucial that every student knows key results, appearing in boxes below.

Consider a column of air. The following discussion applies to the longitudinal sound wave for any gas or liquid. It also applies to the longitudinal sound wave for a solid with a simple change ($B \to Y$, cf. the last LN). However, just to be definite, we will stick with the air case.

We assume that the column of air has a certain length and a cross sectional area A. If it helps you, then we might consider the air as trapped in a long cylindrical tube. You have your friend on the other side of the tube, and you are talking to your friend through the tube. When you speak, the sound propagates through air.

First, suppose that there is no sound, and **the air is in equilibrium**. You slice the air into many identical thin slices, each with cross sectional area A and thickness Δx. For the derivation of the wave equation, it suffices to consider only one such slice! We may find it convenient to assume that this slice is somewhere in the middle of the column of air, while it is actually fine also if the slice ends up at either edge of the air column. In any case, let us imagine that the column of air lies horizontal[1]. The left edge of the slice that we chose—we define its position as x_1. The right edge of the slice that we chose—we define its position as x_2. So, for our thin slice of air that we randomly chose, we have the following.

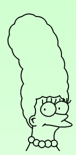
$$x_1 \text{ (left edge of the slice)}, \; x_2 \text{ (right edge of the slice) } \textbf{in equilibrium} \tag{5.2}$$

$$\Delta x = x_2 - x_1 \qquad \Delta x \text{ is very small (very thin slice)} \tag{5.3}$$

Now, consider a **general situation, i.e., a possibly non-equilibrium situation**, in which sound may be propagating. If we are monitoring the same slice of air, then, generally the slice of air is *displaced from* the original position, and it *can be decompressed or compressed* relative to the equilibrium pressure. All of these can be described just by monitoring the position changes of the left edge and the right edge! We define these *position changes* relative to the equilibrium as y_1 and y_2, respectively.

$$x_1 + y_1 \text{ (left edge of the slice)}, \; x_2 + y_2 \text{ (right edge of the slice) } \textbf{in general} \tag{5.4}$$

Notice that x_1, y_1, x_2, y_2 are all measured in the same direction—along the length of the air column. Here is an important connection to Eq. 5.1.



x and D

In wave descriptions, we are now accustomed to writing $D(x, t)$ as the displacement that forms a wave pattern. In the current discussion, the following definitions make the necessary connections.

$$x \equiv x_1 \tag{5.5}$$

$$D \equiv y_1 \tag{5.6}$$

These definitions identify the position of the left edge of the slice, as the position of the slice. Alternatively, we can use mean values $x \equiv \frac{x_1+x_2}{2}$ and $D \equiv \frac{y_1+y_2}{2}$. Yet a third alternative is to use right edges $x \equiv x_2$ and $y \equiv y_2$. All of these three choices are equivalent as their differences are negligibly small in the limit of very thin slice.

[1]Vertical is also just fine. Here we assume horizontal, just to be definite.

Let us consider the volume of our slice of air. In equilibrium its volume is given by

$$V = A(x_2 - x_1) = A\Delta x \qquad \text{from Eqs. 5.2,5.3} \tag{5.7}$$

In general, we will denote its volume as $V + \Delta V$, with ΔV being the change of volume with respective to the equilibrium

$$V + \Delta V = A((x_2 + y_2) - (x_1 + y_1)) = V + A\Delta y \qquad \text{from Eq. 5.4} \tag{5.8}$$

with $\Delta y \equiv y_2 - y_1$. Using the above definition of Eq. 5.6, we can replace y with D, from now on.

$$\Delta V = A\,\Delta D \tag{5.9}$$

Now, using the definition of the bulk modulus $-B\Delta V/V = \Delta P$ (where ΔP is the pressure change relative to the equilibrium), we get

$$\Delta P = -B\frac{\Delta V}{V} = -B\frac{A\,\Delta D}{A\Delta x} \approx -B\frac{\partial D}{\partial x}$$

where in the last step A has been cancelled out, $\Delta x \to 0$ limit is taken, and note is taken of the fact that $D = D(x,t)$. Thus, we get

$$\boxed{\Delta P = -B\frac{\partial D}{\partial x}} \tag{5.10}$$

This is a *great* result. It tells us how to relate the displacement D and the pressure value ΔP, where both quantities are referenced to the equilibrium values.

Now, in order to derive the wave equation, these kinematical considerations are not enough. We need dynamics—Newton's second law. The reason why our thin slice of air experiences any force is due to the two pressure values, P_1 on the left and P_2 on the right. Using our definition of ΔP, we can write[2]

$$P_1 = P + \Delta P_1 \tag{5.11}$$
$$P_2 = P + \Delta P_2 \tag{5.12}$$

The net force that the slice of air experiences is then given by

$$F = (P_1 - P_2)A \tag{5.13}$$

[2]In equilibrium, every part of the medium has the same pressure, which we denote as P.

where the negative sign comes from the fact that the air just on the right side of our thin slice of air exerts pressure to the left. We need to equate this to ma, where $m = \rho V = \rho A \Delta x$ (Eq. 5.3). For a, we must take the second time derivative of the position of the thin slice $x + D$ (cf. the convention Eq. 5.6). Since x is independent of t by definition, and since $D = D(x,t)$, we get $a = \frac{\partial^2 (x+D)}{\partial t^2} = \frac{\partial^2 D}{\partial t^2}$. So, collecting all results, Newton's second law reads

$$\rho A \Delta x \frac{\partial^2 D}{\partial t^2} = (P_1 - P_2)A = (\Delta P_1 - \Delta P_2)A$$

Dividing both sides by $A\Delta x$, and using Eq. 5.10, we get

$$\rho \frac{\partial^2 D}{\partial t^2} = B \frac{\left.\frac{\partial D}{\partial x}\right|_{x=x_2} - \left.\frac{\partial D}{\partial x}\right|_{x=x_1}}{\Delta x}$$

Dividing both sides by B, and defining

$$v = \sqrt{\frac{B}{\rho}} \tag{5.14}$$

we get

$$\frac{1}{v^2} \frac{\partial^2 D}{\partial t^2} = \frac{\left.\frac{\partial D}{\partial x}\right|_{x=x_2} - \left.\frac{\partial D}{\partial x}\right|_{x=x_1}}{\Delta x} \tag{5.15}$$

The quantity on the right hand side is $\frac{\partial^2 D}{\partial x^2}$ in the limit of $\Delta x \to 0$, which is what we had in mind all along, by assuming our thin slice of air to be very thin.

This completes the proof for the wave equation, Eq. 5.1, for the particular case of a longitudinal sound wave in an isotropic medium (gas, liquid, isotropic solid). For a general solid, this derivation is valid, if we merely change B to Y (Young's modulus).

5.4 Implications of the wave equation

We spent quite some time to derive the wave equation, since it is of fundamental importance.

$$\frac{\partial^2 D}{\partial x^2} = \frac{1}{v^2} \frac{\partial^2 D}{\partial t^2} \qquad D = D(x,t),\ v > 0 \tag{5.1}$$

Here are some important properties of this wave equation.

1. This wave equation is valid for various waves such as string wave, sound wave, and light wave (but not for all waves we know).
2. This wave equation is valid for transverse wave (D perpendicular to the direction of the wave propagation) and longitudinal wave (D parallel to the direction of the wave propagation). The speed v can, and generally do, differ for them.
3. This wave equation is valid for general waves, not just for a travelling sinusoidal wave.
   (a) Given any twice-differentiable function, $g(X)$, of variable X, $g(x \pm vt)$ is a solution to the above wave equation.
   (b) Given any two solutions of the above wave equation, say $D_1(x,t)$ and $D_2(x,t)$, the sum of them $aD_1(x,t)+bD_2(x,t)$ is also a solution, where a and b are arbitrary constants.
4. For, and only for, a travelling sinusoidal wave, $D = A\sin(kx - \omega t + \phi)$, the following holds.

$$\omega = vk \tag{4.6}$$

The first two properties have been substantially demonstrated in the previous two sections, while we haven't discussed anything about light yet, and you would have to wait until you take higher level courses to really learn some fundamental physics of light.

The proof of property 3 is easy, and is left for your work.

How about the last property? From the previous lecture, we already discussed this relation for a travelling sinusoidal wave. Note that the reason that this relation $\omega = vk$ is valid for sinusoidal waves, but not for other forms of periodic waves, is a matter of convention: we define k and ω only for sinusoidal waves (see the last LN). In addition, note that $\omega = vk$ is valid only for *travelling* sinusoidal waves; for instance, ω and k are well defined for a standing sinusoidal wave, but $v = 0$ for it!

It might seem redundant to state $\omega = vk$ here, when we have already discussed it in the last lecture. Not so. The point is that the sinusoidal wave is a solution to the wave equation, and ω and k that appear in the definition the sinusoidal wave is related to v that appears in the wave equation by this familiar dispersion relation. To demonstrate this is very easy. For $D(x,t) = A\sin(kx - \omega t + \phi)$, note that $\frac{\partial^2 D}{\partial x^2} = -k^2 D$ and $\frac{\partial^2 D}{\partial t^2} = -\omega^2 D$. Plugging these results into the wave equation, and multiplying both sides by -1, we get

$$k^2 D = \frac{1}{v^2}\omega^2 D.$$

Since this equation must be valid for any arbitrary value of D, we may divide both sides by D, and get $k^2 v^2 = \omega^2$. By definition, we take all of k, v, ω as positive values, and so we get $\omega = vk$, proving that the v value appearing in the wave equation is the same as what we figured out as the wave speed of the travelling sinusoidal wave in the last LN.

5.5 Superposition principle

Whatever goes by a "principle" is to be properly respected, since it is important. Otherwise, we would have given a strong name like it!

Indeed, the superposition principle is one of the defining characteristics of the wave, as opposed to the particle. Waves can be superposed on one another. Particle states cannot, in general.

We have seen the superposition principle, as property 3b, in the list above.

Let us consider the wave equation, Eq. 5.1. We assume that v is a constant, in the sense discussed in Section 5.1. Then, the following **superposition principle** holds.

If $D_1(x,t)$ is a solution to the wave equation, as is D_2, then any linear combination of the two

$$D(x,t) = aD_1(x,t) + bD_2(x,t) \tag{5.16}$$

where a, b are arbitrary constants, is also a solution to the wave equation.

Rather than thinking of this principle as a physical principle of real states, one must think of it as a physical principle of virtual states. The reason is illustrated by following examples.

Let us consider $D_1 = A\sin(kx - \omega t)$, which is a solution of Eq. 5.1, as long as $\omega = vk$. Now, if we take $D_2 = -D_1$ (which is a solution to Eq. 5.1 as well), and

$a = 1, b = 1$, then we get $D = 0$. This *null* wave is indeed a solution to the wave equation! Now, if you were to interpret this physically, then you might think that there is a wave D_1 and there is another wave $D_2 = -D_1$, and they completely destroy each other to give the total displacement $D = 0$. Clearly, there is positive energy in either D_1 and D_2 states, since the mechanical energy stored in a sinusoidal wave is the sum of all energies of the simple harmonic motions involved and since the energy of a SHO is non-negative. Clearly, the energy for $D = 0$ is zero! So, there is an energy non-conservation. This should not happen in physics!

Well, this *can* happen in physics, if one is considering only virtual states, i.e., those states that are possible but not necessarily realized. If one thinks of a null wave $D(x,t) \equiv 0$ as a superposition of two waves that completely compensate each other, then the null wave can be regarded as real, but the other two waves (D_1 and $-D_1$) are definitely not real—they are imagined mathematically but not realized, i.e., they are virtual.

Let us now consider another example. Let us suppose that two waves, D_1 and D_2, *exist* separately at some time (say $t = t_s$), i.e. they are separated in space, and then come together at some other time, say $t = t_t$. At *any* time, the total wave is given by $D = D_1(x,t) + D_2(x,t)$. However, in this case, the energy of D_1 at time t_s plus the energy of D_2 at time t_s is equal to the total energy of the wave at *any* time, including $t = t_t$. In this second example, all of D_1, D_2, and D are real.

Generally, the latter situation is referred to as **interference**.

Note that the superposition principle is more fundamental; it holds whether or not all wave states under consideration are real or not.

Finally, there is one subtlety to note regarding the superposition principle. As presented above, the superposition principle is easily formulated under the assumption that v is really a constant. In reality, this is not quite true; v tends to be dependent on k (as in prism or rainbow) as well as on whether or not the wave is transverse or longitudinal. So, then the superposition principle breaks down? Not quite. Even when different speeds are involved, low amplitude waves tend to display the superposition principle to a good approximation, not because they satisfy one wave equation when combined (they cannot), but because they simply do not interact with one another (the presence of one wave has no influence on the other wave). However, if waves with large amplitudes mix, then the superposition principle breaks down, since constituent waves interact; this is a regime where the simple Hooke's law physics is not sufficient, and we call this regime a "non-linear regime." In this course, we do not concern ourselves with the non-linear regime.