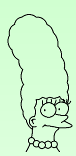
Notes for Lecture 2

Simple harmonic oscillation

2.1 Terms of significance

We learned that the equation of motion for a simple harmonic oscillation is given by $\ddot{x} = -\omega^2 x$ and its general solution is given by $x = A\cos(\omega t + \phi)$. While other equivalent forms exist, as we learned in the last lecture, we will stick to this cosine form, whenever possible.



Phase, amplitude, frequency, period

The quantity $\omega t + \phi$ is called **phase**. ϕ is the **initial phase**, since it is the phase at $t = 0$. A is called the **amplitude**. By convention, both A and ω are taken to be non-negative (or positive, since $A = 0$ or $\omega = 0$ represents a static state).

$$T = \frac{2\pi}{\omega} \qquad \textbf{period}\text{; unit = sec} \tag{2.1}$$

$$f = \nu = \frac{1}{T} = \frac{\omega}{2\pi} \qquad \textbf{linear frequency}\text{; unit = 1/s = Hz} \tag{2.2}$$

$$\omega = 2\pi f = \frac{2\pi}{T} \qquad \textbf{angular frequency}\text{; unit = Hz or rad/s} \tag{2.3}$$

2.2 Velocity, acceleration

Now that we have obtained the general solution for x, we can proceed to calculate v and a.

$$v = \dot{x} = -A\omega \sin(\omega t + \phi) \tag{2.4}$$
$$a = \dot{v} = -A\omega^2 \cos(\omega t + \phi) \tag{2.5}$$

We see that when $|x|$ vanishes, $|v|$ maximizes, and vice versa. On the other hand, note that $a = -\omega^2 x$ (as expected from the EOM), and so $|a|$ and $|x|$ go up and down at the same time.

It is instructive to calculate the maximum *speed* v_{max} and the maximum *magnitude* of acceleration a_{max}.

$$v_{max} \equiv \max(|v|) = A\omega \tag{2.6}$$
$$a_{max} \equiv \max(|a|) = A\omega^2 \tag{2.7}$$

2.3 "Disappearing" gravity

The vertical spring problem (Figure T14-3) is kind of interesting. Apparently, this problem is very confusing. Let us clear up any confusion.

We use the time-honored strategy: **divide and conquer**.

Let us ignore the spring, and consider the gravity only. Then, force = $F_g = mg$ (down is positive, by convention of Figure T14-3). Next, we turn *off* the gravity, and consider the spring only. Then, force = $F_s = -k(x_0 + x)$. It is crucial to note that without the gravity, $x_0 + x$, *not* x, measures the distance from the equilibrium length with no gravity.

So, by dividing the problem into two separate problems each containing only one force, we can find each force without any ambiguity.

Now, what will happen if both forces are turned on? We simply need to add the two forces!

So, the net force in the original problem is given by

$$F = F_g + F_s = mg - k(x_0 + x) \tag{2.8}$$

This expression makes a perfect sense: there is gravity and there is spring force.

Now, the magical part: we use the fact that $kx_0 = mg$ (cf. Eq. 1.3), and re-write the above equation as

$$F = F_g + F_s = -kx \tag{2.9}$$

So, the gravity "disappeared"!

Note that in the above discussion, $U_g = -mgx$, not mgx, since x has been defined positive *downward* (Fig. T14-3).

What's in a name?

In the above discussion, $F = -kx$, is the net force, *gravity plus spring force*, not just the spring force. This may be hard to believe, since somehow we automatically recognize, without thinking, $-kx$ as the spring force! **Do not automatically assume that the expression $-kx$ is the spring force!** Name (like x) is *not* important. What is important is what the name *means*.

So, how do we *really* know that $-kx$ is the net force? It does satisfy the two requirements: (1) zero at $x = 0$, and (2) the x-slope must be $-k$.

The spring force in this case is *not* $-kx$ but $-kx - mg$ (net force minus the gravity)!

The conclusion is then that even though $F = -kx$ looks like a simple spring force, it really is not in the current case. It is an "effective" spring force due to the combined effect of the gravity and the spring force.

2.4 Energy conservation

For a spring, either vertical or horizontal, the simple EOM, $F = -kx$ is now established. Thus, it follows that the total mechanical energy is conserved.

$$E = K + U = \frac{1}{2}mv^2 + \frac{1}{2}kx^2 = \text{constant} \tag{2.10}$$

Another view of the "disappearing" gravity

While this may seem a bit magical, its origin can be seen in another way as follows. The short answer is: because the gravitational potential energy (for surface gravity) is a linear function of x, its only role is to re-define the equilibrium position, while it has no role in determining the nature of the motion around the equilibrium position. Here is a longer answer, more specific to the above problem.

As mentioned in the last lecture, the oscillation around a stable equilibrium *is* a SHO[a]. A SHO is completely determined by ω, assuming that we know the equilibrium position. Now, $\omega = \sqrt{k/m}$ and k is, in more general terms, given by

$$k = \left.\frac{d^2U}{dx^2}\right|_{x=x_{eq}} \tag{2.11}$$

where U is the potential energy. Note that the gravitational potential energy

$$U_g(x) = -mgx \tag{2.12}$$

does not contribute to k at all! On the other hand, the spring potential energy

$$U_s(x) = \frac{1}{2}k(x+x_0)^2 \tag{2.13}$$

does contribute to the second derivative value, k! Namely, for the current problem

$$\frac{d^2U}{dx^2} = \frac{d^2(U_g+U_s)}{dx^2} = k \qquad \text{for any } x\text{, actually, including } x = x_{eq} \tag{2.14}$$

Namely, the spring constant is unchanged, and there is no role of g other than re-defining the equilibrium position. One can also show explicitly (using $mg = kx_0$ (Eq. 1.3)) that

$$U(x) = U_g(x) + U_s(x) = \frac{1}{2}kx^2 + \frac{1}{2}kx_0^2 \tag{2.15}$$

[a] In general, this is true only for small deviation from the equilibrium.

Note that this energy conservation is the consequence of the EOM[1]. Indeed, all of the so-called Newtonian mechanics comes down to $\vec{F} = m\vec{a}$, from which every result of Newtonian mechanics can be derived[2].

If we consider the motion of mass on spring, then it is clear that when $x = \pm A$ (**turning points**), $v = \dot{x} = 0$. So, $E = \frac{1}{2}kA^2$. Also, when $x = 0$, $U = 0$, and $K = \frac{1}{2}mv_{max}^2 = \frac{1}{2}m\omega^2 A^2$ (from Eq. 2.6). So, we get

$$E = K + U = \frac{1}{2}mv^2 + \frac{1}{2}kx^2 = \frac{1}{2}kA^2 = \frac{1}{2}m\omega^2 A^2 \qquad (2.16)$$

While what we have showed so far is proof enough, we can also demonstrate the energy conservation *explicitly*, using $x = A\cos(\omega t + \phi)$. By plugging this expression for x and $v = \dot{x} = -A\omega\sin(\omega t + \phi)$ into Eq. 2.10, and using $\omega^2 = k/m$, it is straightforward to show (as done in class) that $E = \frac{1}{2}kA^2 = \frac{1}{2}m\omega^2 A^2$.

Note that

$$K = \frac{1}{2}m\omega^2 A^2 \sin^2(\omega t + \phi) = E\,\sin^2(\omega t + \phi) \qquad (2.17)$$

$$U = \frac{1}{2}kA^2\,\cos^2(\omega t + \phi) = E\,\cos^2(\omega t + \phi) \qquad (2.18)$$

It is left for your exercise to show that[3]

$$\frac{1}{T}\int_0^T dt\,\sin^2(\omega t + \phi) = \frac{1}{T}\int_0^T dt\,\cos^2(\omega t + \phi) = \frac{1}{2} \qquad T = 2\pi/\omega \qquad (2.19)$$

So, we get

$$\langle K\rangle \equiv \frac{1}{T}\int_0^T dt\,K = \frac{E}{2} \qquad \text{average kinetic energy} \qquad (2.20)$$

$$\langle U\rangle \equiv \frac{1}{T}\int_0^T dt\,U = \frac{E}{2} \qquad \text{average potential energy} \qquad (2.21)$$

That $\langle K\rangle$ and $\langle U\rangle$ are equal to each other can be seen by plotting $\sin^2(\omega t + \phi)$ and $\cos^2(\omega t + \phi)$ functions, as done in class.

[1] It is a direct consequence of applying the work-energy theorem (cf. `https://griffin.ucsc.edu/forum/question/154/work-energy-theorem`) in the case of a conservative force. The work-energy theorem is a direct consequence of $\vec{F} = m\vec{a}$.

[2] An exception is Newton's third law. However, in modern view, Newton's third law can be considered a simple input arising from Nature's fundamental laws of forces.

[3] Hint: use $\sin^2\theta = \frac{1-\cos(2\theta)}{2}$ and $\cos^2\theta = \frac{1+\cos(2\theta)}{2}$.

2.5 Uniform circular motion and SHM

Consider a uniform circular motion (UCM) in an xy plane, with radius A. This means that the angular velocity $\dot{\theta} = \omega =$ constant. Then, we get $\theta = \omega t + \phi$, where we defined $\phi \equiv \theta(t = 0)$. Then, we get (by using $x = r\cos\theta$ and $y = r\sin\theta$ with $r = A$)

$$x = A\cos(\omega t + \phi) \tag{2.22}$$

$$y = A\sin(\omega t + \phi) \tag{2.23}$$

We recognize that both these are SHM solutions (Eqs. 1.6 and 1.7). This means that the projection of a UCM onto the x axis or the y axis is a SHM. This can be extended to any axis at an arbitrary angle to the x axis.

We see that the radius of a UCM corresponds to A of the SHM, and the **angle** θ of the UCM corresponds to the **phase** $\omega t + \phi$ of the SHM, with ϕ being the initial angle.