Due Jan. 30, Thursday

Problem 1 (80 points) Consider a spring of equilibrium length L, lying horizontally in a frictionless trough. The spring has cross sectional area S perpendicular to its length. The trough constrains the motion of the spring so that any wave propagating along the spring is a longitudinal wave. Here, our goal is to find the wave equation for such a longitudinal wave. The spring constant for this spring is k. The spring has a uniform linear mass density, μ, in equilibrium: $\mu = M/L$, where M is the total mass of the spring.

To describe this longitudinal wave, we can start by doing a Newtonian mechanics calculation[1]. Key variables to pay attention to are x and D: x is the position of the points of the spring *in equilibrium*, and $D(x,t)$ is the displacement of each such point relative to its fixed equilibrium position, x. So, x is *not* the position of the "particle" (defined rigorously in the next paragraph) in the presence of wave, but just an "index" for each particle, as in all wave descriptions; the position is given by $x + D(x,t)$, where x is t-independent.

Consider a very small positive value of Δx such that $\Delta x = L/N$ where N is a large positive integer. We can consider that the spring is divided into small segments of equal length Δx in equilibrium. Each segment is what we can treat as a "particle" within Newtonian mechanics. For this reason, we shall call each segment a "particle spring." Let us define x as the position of the left end of a particle spring. Each particle spring is elongated or compressed as wave propagates, and such elongation or compression results in finite values of $D(x,t)$ (displacement at left end) and $D(x+\Delta x,t)$ (displacement at right end). It is important to note that within each particle spring, the spring is *not* uniformly compressed or elongated, in general, and Hooke's law *cannot* be applied to such a non-uniformly deformed spring as a whole.

Because of the last property, we must divide each particle spring of ours further into many littler springs! For this, we take $\varepsilon = \Delta x/N_2$. Here, N_2 is a large enough positive integer so that each littler spring of equilibrium length ε, which we can call a "nano spring," can be considered as *uniformly* compressed or elongated, always. Therefore, in general, when wave propagates, we can apply Hooke's law for, and *only for*, each nano spring.

Mathematically, either Δx or ε is equivalent to "the infinitesimal" in differential calculus. Note that, in this problem, k is used for the spring constant of the entire spring, and is *not* used for wave number.

(a) Find the spring constant of each particle spring (k_p) in terms of k and N. [Hint: This is a spring in series problem, discussed in class. Consider a situation where the length of the spring changes, $L \to L + \Delta L$, uniformly, and

[1] While this calculation has an essential similarity to that for a sound wave described in Section 5.3, you do *not* need to understand that section to do this problem.

consider Newton's third law and Hooke's law (which *is* applicable if the spring is uniformly compressed or elongated, *not* carrying a wave).]

(b) Find the spring constant of each nano spring (k_{nano}), in terms of k, N, and N_2.

(c) Consider a nano spring whose left end is at x and whose right end is at $x+\varepsilon$ in equilibrium. $D(x,t)$ and $D(x+\varepsilon,t)$ cause a finite spring force exerted by this spring. Prove that the spring force exerted by this nano spring at left end is proportional to $\frac{\partial D}{\partial x}$, and find the proportionality constant in terms of k and L.

(d) Now consider a nano spring whose left end is at $x+\Delta x-\varepsilon$ and whose right end is at $x+\Delta x$ in equilibrium. Prove that the spring force exerted by this nano spring at right end is proportional to $\frac{\partial D}{\partial x}\Big|_{x+\Delta x}$, and find the proportionality constant in terms of k and L.

(e) From the answers of the previous two parts and Newton's third law, it is now possible to calculate the net force acting *on* the particle spring at index x, and use Newton's second law to set up the equation of motion for the particle. Find the equation of motion. Your answer must involve the following symbols only: D, t, x, μ, k, L.

(f) Comparing the equation of motion that you obtained in the previous part with Eq. 5.1, identify the wave speed in this case. Check the physical dimension of your answer.

(g) Young's modulus is defined as $Y = \frac{F/S}{|\Delta L|/L}$, where $F = k|\Delta L|$ is the applied force, and ΔL is the spring contraction or elongation. Find Y in terms of k, L, and S.

(h) Express the speed of wave in terms of Y and ρ, where ρ is the volume mass density $\rho = \mu/S$. Compare your answer with Eq. 4.21.

Problem 2 (40 points) A string with mass M and length L is hanging from the ceiling.

(a) The string is at rest. Let us define the coordinate from top to bottom as x ($x=0$ at top and $x=L$ at bottom). Find the tension in the string as a function of x, M, L, and g.

(b) Consider a transverse string wave generated on this string. Show that the speed of the wave, v, is x dependent and find v as a function of x, M, L, and g (some of these symbols may not appear in the answer).

(c) You push the bottom end of the string (lightly) in the horizontal direction. Assuming that there is no loss of energy (no damping). Will the wave propagate all the way to the top? Explain your answer briefly.

(d) You disturb the string very near the top, by pushing the very near top part horizontally. Again, assume no damping. How long will it take for this disturbance to reach the bottom of the string? Your answer must be expressed as a function of L, g, and M (some of these symbols may not appear in the answer).