Notes for Lecture 8

Waves, Sound

We have derived the wave equation in the last lecture, and it was good to do so. My sense is that every student will be able to understand it, if the derivation is studied quietly.

From the wave equation, we could precisely define what **the superposition principle** means. If this is unclear, please go back to LN 7 and read the last section! The subtle differences between superposition and interference have been discussed there as well. In this LN, we discuss some facts that seem to be slightly challenging to students in general.

8.1 Frequency

The box below summarizes a very useful fact to know. You can get a lot of mileage out of knowing this fact!

But, why is this fact true? It is because, when a wave propagates, what is happening is that a group of atoms shakes another group of atoms adjacent to them. Whether this "shaking" is happening between your hand and a string, between adjacent segments of a string, at the joint of two strings of different thickness, between air molecules and the molecules in your ear drum, or between water molecules and air molecules at the ocean surface, this shaking is coordinated, and the frequency is preserved.

Importance of frequency

When a wave propagates through various different media, through reflection, refraction, transmission, diffraction, etc., there is one thing that never[a] changes: frequency. Here, by *different media*, we mean not only different materials, but also the same material kept in different conditions, e.g., temperature, string tension (for string wave), pressure, and density.

[a] Please read the next box!

Never say never!

A wise student would know that the word "never" never really means "never." Like in every usage of "never," what is really meant by the footnoted word "never" in the previous box means more "almost never" than never. But a strong effect that accompanies "never" was probably of some value. Now, here is a more refined discussion for you, who strives for sophistication.

Let us recall that a medium can generally display "non-linearity" (e.g., page 4 of LN 1 and the box in page 4 of LN 5). The non-linearity lead to many very interesting, and sometimes very basic, phenomena (chaos, higher harmonic generations with laser, thermal expansion/contraction of materials, etc.). It also leads to the breakdown of the superposition principle (Section 7.3). However, for many physical phenomena, linear physics is still dominant. For instance, *even when the dispersion relation (page 4 of LN 5) is non-linear, the superposition principle can be taken to be valid, approximately.* This is what will go on for most topics that you learn in the undergraduate curriculum (exceptions include the chaos physics that you may learn in classical mechanics course and various things that you may learn in solid state physics). *A common and important non-linear effect is the frequency change of a portion of a wave when it propagates through a non-linear medium.* This however is a small effect in most cases, for which our use of the word "never" in the previous box is thus justified to a good approximation.

8.2 Energy of wave

Consider a sinusoidal wave. What is the energy contained in the wave? This is an easy thing to figure out, if you notice that at each point, the wave medium is doing a simple harmonic motion. Using this fact, the energy can be derived in a straightforward manner.

For a travelling sinusoidal wave, the result is Eq. T15-5 (where vt is the length of the medium; also note that ρS can be identified as the linear density μ, especially useful for a string wave). The energy transmitted per time is power, as given by Eq. T15-6. The intensity is then given by the power per area, Eq. T15-7.

For a wave generated by a constant power source, the intensity for a travelling sinusoidal wave (i.e., a **plane wave**; $D = A\sin(kx - \omega t + \phi)$) is constant, since the wave propagates only in one direction.

This is not the case for common waves in three dimensions. The light from a light bulb propagates in all three directions. This proper wave to consider in this case is a **spherical wave**: $D = \frac{A}{r}\sin(kr - \omega t + \phi)$. Note that a spherical wave will look like a plane wave if it is observed in a small region of space, just as we can consider the earth flat in our everyday lives. Then, one can see that the "effective plane wave amplitude" for a spherical wave is A/r. Using this in Eq. T15-7 (i.e, by replacing A with A/r), one sees that $I \propto r^{-2}$. This is an inverse square law for the spherical wave. It arises from the fact that the energy deposited at the center to cause a spherical wave must be conserved as the spherical wave propagates. As the energy is spread on the area of $4\pi r^2$ as it propagates, and as the energy is proportional to A^2 in the plane-wave case, it follows that the effective plane wave amplitude of the spherical wave must diminish as $1/r$.

8.3 The nature of the SHM participating in a wave

Consider a particle with mass $m = \mu dx$ that goes through a SHM in a travelling sinusoidal wave:

$$D(x,t) = A\sin(kx - \omega t + \phi) \tag{5.1}$$

The energy for this SHM is given by

$$dE = \frac{1}{2}m\omega^2 A^2 = \frac{1}{2}\mu\omega^2 A^2 dx \tag{8.1}$$

This is the first step to calculate the total energy contained in the wave $E = \int dE$.

As discussed in the lecture, this energy dE can be considered as $\frac{1}{2}\kappa A^2$ with κ being a Hooke's law spring constant: $\kappa = m\omega^2 = \mu dx\omega^2$. What is the origin of κ? Recall the wave equation

$$\frac{\partial^2 D}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 D}{\partial t^2} \qquad D = D(x,t) \tag{7.1}$$

Plugging in the above solution to the left hand side, we see that this wave equation becomes

$$-k^2 D = \frac{1}{v^2}\frac{\partial^2 D}{\partial t^2} \tag{8.2}$$

$$\therefore \ddot{D} = -(kv)^2 D \qquad x \text{ fixed} \tag{8.3}$$

As expected this is a SHO EOM for the displacement D of the particle at x with ω given by vk. Therefore, one can assign $\kappa = m\omega^2 = \mu dx\omega^2$ for this EOM.

However, here is a more physical question. **Is κ an inherent natural property of the medium? The answer is no! It is rather a property determined by the external driving frequency ω.** Here is what I mean – for a fixed dx value and a fixed μ value, κ is entirely determined by ω, which is determined by the source of the wave! So, it would be a mistake to think that κ is somehow fixed by a natural property of the particle in question and its interaction with neighbors. This is different from the SHO problem that we discussed in Lectures 1-4, where the Hooke's law existed prior to any consideration. In general, in a sinusoidal wave phenomenon, all there is is a synchronized set of SHO motions of all particles. The appearance of Hooke's law is less important: Hooke's law is dependent on the wave pattern itself! This is because a wave phenomenon is a collective emergent phenomenon (Section 4.2).

If this is clear, it is excellent. If this is confusing, however, I understand. There is one quick remedy: do *not* think in terms of Hooke's law κ when a wave and its associated SHMs are considered!

In any case, note that a general wave medium may not have a natural frequency. An entirely free string is an example. It will transmit wave of any frequency. Unrestricted air is another example. Sound of any frequency is transmitted through it. However, if boundary conditions are imposed, then natural frequencies can develop. For example, a string can be clamped at both ends, like a guitar string, and suddenly there are only those discrete wave lengths and the corresponding frequencies that are allowed! When an external perturbation excites the guitar string with ω, the guitar string will respond most strongly at those natural frequencies. The string will respond to non-resonant frequencies also, but the response will be weak and will soon die out,

due to the damping that exists for any real system. For this reason, the system can be considered to respond only to a discrete set of resonant frequencies (cf. the last question for the practice exam for midterm 1). Another example is air trapped in a container like a wind instrument. In these cases where a discrete set of resonant frequencies exist, one can think that κ values (which will be form a discrete set) reflect the inherent property of the medium, arising from its boundary conditions.

8.4 Reflection – boundary conditions

Please read Section T15-7. We have seen the demo for Figure T15-18.

The considerations in Section T15-7 fall into what we call the **boundary conditions**. For a fixed end of a string, D is forced to be zero: that is a boundary condition imposed on D. Now suppose a string wave is reflected off of the fixed end, and suppose that $D_1(x,t)$ is the original wave that existed far away from the fixed end and propagated to the fixed end. At any time, we can consider the total wave $D(x,t)$ as the superposition of two waves: $D = D_1(x,t) + D_2(x,t)$, where D_1 is the original wave, which disappears when the wave is reflected, and D_2 is the reflected wave, which appears when the wave is reflected. The boundary condition implies that at the fixed end, $D_1 + D_2 = 0$. This happens only if there is a π **phase shift**[1] from D_1 to D_2! This is because a π phase shift will introduce a negative sign, no matter how you write a sinusoidal wave[2]: $\sin(kx - \omega t + \phi + \pi) = -\sin(kx - \omega t + \phi)$ and $\cos(kx - \omega t + \phi + \pi) = -\cos(kx - \omega t + \phi)$.

In contrast, a free end of a string wave does not impose any boundary condition. And a bit of physical reasoning will convince you that there must be no phase shift at all.

How about sound waves? The same considerations apply. Consider an air trapped in a tube with an open end and a close end. An open end is a free end for the displacement D. A closed end is a fixed end for the displacement. At the closed end, the boundary condition $D = 0$ applies, and there is a π phase shift.

Now, there is more story to tell in the case of this sound wave example. How about the pressure ΔP (Eq. 7.10)? Physically speaking the closed end of the tube is a *free end* for pressure, since the closed end can take any pressure. The open end of the tube is a *fixed end* for pressure, $\Delta P = 0$, since it is where the air mass inside the tube must negotiate with the large air mass outside the tube – the pressure must

[1] Actually, $-\pi$ or 3π etc. will do fine as well, but any angle/phase is equivalent modulo 2π.

[2] Or, no matter how many sinusoidal waves that you sum up.

come quickly to that of the atmospheric pressure! So, when the sound wave is viewed as the oscillation of ΔP, instead of D, then the wave goes through a π phase shift when the sound is reflected at an open end! And, no phase shift at a closed end.

Please study Eq. 7.10 carefully, to convince yourself that these two views of sound waves, one in terms of D and the other in terms of ΔP, are indeed mathematically compatible (hint: dx will change sign when a wave is reflected).