Notes for Lecture 7

Waves, Sound

7.1 Wave equation

For many, but not all, waves, the following wave equation holds. For instance, it holds for string wave, sound wave, and light.

$$\frac{\partial^2 D}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 D}{\partial t^2} \qquad D = D(x,t) \tag{7.1}$$

For the purpose of this course, we can call this *the* wave equation. But, you can keep in the back of your mind that there might be other wave equations in physics – you will encounter them in advanced courses. One thing that you can already know is that many qualitative features of wave are shared by different waves satisfying different wave equations – the most important property is perhaps the superposition principle, which we will discuss at the end of this LN.

The above wave equation describes a wave propagating at speed v. The above equation is valid even when v is not a constant, in the sense that it depends on k. Since $\omega = vk$, we see that when v is dependent on k, $\omega(k)$ is a non-linear function of k. In such a case, the medium is said to be **dispersive**. Generally, the function $\omega(k)$ is defined as the **dispersion relation**.

7.2 Wave equation for longitudinal sound

It seems useful to derive the wave equation for this particular problem, since it illustrates many points. While every student should follow this derivation, it is not required that she/he knows to how to reproduce the derivation. However, it is crucial that every student knows key results, appearing in boxes below.

Consider a column of air. The following discussion applies to the longitudinal sound wave for any gas or liquid. It also applies to the longitudinal sound wave for a solid with a simple change ($B \to Y$, cf. Eq. 6.4). However, just to be definite, we will stick with the air case.

We assume that the column of air has a certain length and a cross sectional area A. If it helps you, then we might consider the air as trapped in a long cylindrical tube. You have your friend on the other side of the tube, and you are talking to your friend through the tube. When you speak, the sound propagates through air.

First, suppose that there is no sound, and **the air is in equilibrium**. You slice the air into many identical thin slices, each with cross sectional area A and thickness Δx. For the derivation of the wave equation, it suffices to consider only one such slice! We may find it convenient to assume that this slice is somewhere in the middle of the column of air, while it is actually fine also if the slice ends up at either edge of the air column. In any case, let us imagine that the column of air lies horizontal[1]. The left edge of the slice that we chose – we define its position as x_1. The right edge of the slice that we chose – we define its position as x_2. So, for our thin slice of air that we randomly chose, we have the following.

$$x_1 \text{ (left edge of the slice), } x_2 \text{ (right edge of the slice) } \textbf{in equilibrium} \qquad (7.2)$$

$$\Delta x = x_2 - x_1 \qquad \Delta x \text{ is very small (very thin slice)} \qquad (7.3)$$

Now, consider a **general situation, i.e., a possibly non-equilibrium situation**, in which sound may be propagating. If we are monitoring the same slice of air, then, generally the slice of air is *displaced from* the original position, and it *can be decompressed or compressed* relative to the equilibrium pressure. All of these can be described just by monitoring the position changes of the left edge and the right edge! We define these *position changes* relative to the equilibrium as y_1 and y_2, respectively.

$$x_1 + y_1 \text{ (left edge of the slice), } x_2 + y_2 \text{ (right edge of the slice) } \textbf{in general} \qquad (7.4)$$

Notice that x_1, y_1, x_2, y_2 are all measured in the same direction – along the length of the air column. But, here is the important connection to what we have been doing in previous classes, and to Eq. 7.1.

[1] Vertical is also just fine. Here we assume horizontal, just to be definite.

x and D

In wave descriptions, we are now accustomed to writing $D(x,t)$ as the displacement that forms a wave pattern. In the current discussion, the following definitions make the necessary connections.

$$x \equiv x_1 \tag{7.5}$$

$$D \equiv y_1 \tag{7.6}$$

These definitions identify the position of the left edge of the slice, as the position of the slice. Alternatively, we can use mean values $x \equiv \frac{x_1+x_2}{2}$ and $D \equiv \frac{y_1+y_2}{2}$. Yet a third alternative is to use right edges $x \equiv x_2$ and $y \equiv y_2$. All of these three choices are equivalent[a] as their differences are negligibly small in the limit of very thin slice.

[a] I have a slight change of heart here, in comparison to the discussion during the class. Here, I take the left edge convention, while in class I used the mean value convention. The math is a tad simpler if we use the left edge convention.

Let us consider the volume of our slice of air. In equilibrium its volume is given by

$$V = A(x_2 - x_1) = A\Delta x \qquad \text{from Eqs. 7.2,7.3} \tag{7.7}$$

In general, we will denote its volume as $V + \Delta V$, with ΔV being the change of volume with respective to the equilibrium

$$V + \Delta V = A((x_2 + y_2) - (x_1 + y_1)) = V + A\Delta y \qquad \text{from Eq. 7.4} \tag{7.8}$$

with $\Delta y \equiv y_2 - y_1$. Using the above definition of Eq. 7.6, we can replace y with D, from now on.

$$\Delta V = A\,\Delta D \tag{7.9}$$

Now, using the definition of the bulk modulus $-B\Delta V/V = \Delta P$ (where ΔP is the pressure change relative to the equilibrium), we get

$$\Delta P = -B\frac{\Delta V}{V} = -B\frac{A\,\Delta D}{A\Delta x} \approx -B\frac{\partial D}{\partial x}$$

where in the last step A has been cancelled out, $\Delta x \to 0$ limit is taken, and note is taken of the fact that $D = D(x,t)$. Thus, we get

$$\Delta P = -B\frac{\partial D}{\partial x} \tag{7.10}$$

This is a *great* result. It tells us how to relate the displacement D and the pressure value ΔP, where both quantities are referenced to the equilibrium values.

Now, in order to derive the wave equation, these kinematical considerations are not enough. We need dynamics – Newton's second law. The reason why our thin slice of air experiences any force is due to the two pressure values, P_1 on the left and P_2 on the right. Using our definition of ΔP, we can write[2]

$$P_1 = P + \Delta P_1 \tag{7.11}$$

$$P_2 = P + \Delta P_2 \tag{7.12}$$

The net force that the slice of air experiences is then given by

$$F = (P_1 - P_2)A \tag{7.13}$$

where the negative sign comes from the fact that the air just on the right side of our thin slice of air exerts pressure to the left. We need to equate this to ma, where $m = \rho V = \rho A \Delta x$ (Eq. 7.3). For a, we must take the second time derivative of the position of the thin slice $x + D$ (cf. the convention Eq. 7.6). Since x is independent of t by definition, and since $D = D(x,t)$, we get $a = \frac{\partial^2 (x+D)}{\partial t^2} = \frac{\partial^2 D}{\partial t^2}$. So, collecting all results, Newton's second law reads

$$\rho A \Delta x \frac{\partial^2 D}{\partial t^2} = (P_1 - P_2)A = (\Delta P_1 - \Delta P_2)A$$

Dividing both sides by $A\Delta x$, and using Eq. 7.10, we get

$$\rho \frac{\partial^2 D}{\partial t^2} = B\frac{\left.\frac{\partial D}{\partial x}\right|_{x=x_2} - \left.\frac{\partial D}{\partial x}\right|_{x=x_1}}{\Delta x}$$

Dividing both sides by B, and defining

$$v = \sqrt{\frac{B}{\rho}} \tag{7.14}$$

[2]In equilibrium, every part of the medium has the same pressure, which we denote as P.

we get

$$\frac{1}{v^2}\frac{\partial^2 D}{\partial t^2} = \frac{\left.\frac{\partial D}{\partial x}\right|_{x=x_2} - \left.\frac{\partial D}{\partial x}\right|_{x=x_1}}{\Delta x} \tag{7.15}$$

The quantity on the right hand side is $\frac{\partial^2 D}{\partial x^2}$ in the limit of $\Delta x \to 0$, which is what we had in mind all along, by assuming our thin slice of air to be very thin.

This completes the proof for the wave equation, Eq. 7.1, for the particular case of a longitudinal sound wave in an isotropic medium (gas, liquid, isotropic solid). For a general solid, this derivation is valid, if we merely change B to Y (Young's modulus).

Note that in the string wave case, an analogous derivation has been given in the book (Section T15-5).

7.3 Superposition principle

Whatever goes by a "principle" is to be properly respected, since it is important. Otherwise, we would have given a strong name like it!

Indeed, the superposition principle is one of the defining characteristics of the wave, as opposed to the particle. Waves can be superposed on one another. Particle states cannot, in general.

The superposition principle must be understood mathematically.

Let us consider the wave equation, Eq. 7.1. We assume that v is a constant, in the sense discussed in Section 7.1. Then, the following **superposition principle** holds.

If $D_1(x,t)$ is a solution to the wave equation, as is D_2, then any linear combination of the two

$$D(x,t) = aD_1(x,t) + bD_2(x,t) \tag{7.16}$$

where a, b are arbitrary constants, is also a solution to the wave equation.

The proof is left for your exercise (Hint: it is simple if you use the property $\frac{\partial^2(aD_1(x,t)+bD_2(x,t))}{\partial x^2} = a\frac{\partial^2 D_1}{\partial x^2} + b\frac{\partial^2 D_2}{\partial x^2}$, and similarly for the second derivative for time.)

The reason why this principle must be taken as a mathematical principle is illustrated best by an example.

Let us consider $D_1 = A\sin(kx - \omega t)$, which is a solution of Eq. 7.1, as long as $\omega = vk$. (The proof of this last statement is left for your exercise!) Now, if we take $D_2 = -D_1$ (which is a solution to Eq. 7.1 as well), and $a = 1, b = 1$, then we get $D = 0$. This *null* wave is indeed a solution to the wave equation! Now, if you were to interpret this physically, then you might think that there is a wave D_1 and there is another wave $D_2 = -D_1$, and they completely destroy each other to give the total displacement $D = 0$. Clearly, there is positive energy in either D_1 and D_2 states, since the mechanical energy stored in a sinusoidal wave is the sum of all energies of the simple harmonic motions involved and since the energy of a SHO is non-negative. Clearly, the energy for $D = 0$ is zero! So, there is an energy non-conservation. This should not happen in physics!

Let us now consider another example. Let us suppose that two waves, D_1 and D_2, *exist* separately at some time (say $t = t_s$), i.e. they are separated in space, and then come together at some other time, say $t = t_t$. At *any* time, the total wave is given by $D = D_1(x,t) + D_2(x,t)$. However, in this case, the energy of D_1 at time t_s plus the energy of D_2 at time t_s is equal to the total energy of the wave at *any* time, including $t = t_t$.

Generally, the latter more physical situation is referred to as **interference**. There is one subtlety to note. Generally, the superposition principle applies only to the non-dispersive medium (v is independent of k), or "linear medium" in the sense that Hooke's law leads to k independence of v. However, note that the superposition principle will hold no matter what, if only a single wavenumber/frequency is considered (monochromatic wave).