Notes for Lecture 5

Waves

In the previous lecture, we touched upon the most generic two features of a wave. They can be also summarized this way: a wave is an *emergent property* of many particles. What this means is that when many particles act in coordination, a wave pattern can emerge, and this wave property is distinct from any single particle property. In the mundane example of a wave in a baseball field, that wave is nothing you can attribute to a single person; rather, it represents a group psyche the emerges beyond an individual.

5.1 Travelling sinusoidal wave

Not all waves are **sinusoidal** (or, in another word, **harmonic**). Nevertheless, it is very convenient to consider sinusoidal waves, first. The reason? Due to the Fourier theorem, pretty much all wave forms can be expressed as a linear combination of sinusoidal waves.

A travelling sinusoidal wave can be written down as

$$D(x,t) = A\sin(kx - \omega t + \phi) \qquad A \geq 0, \quad \omega \geq 0 \tag{5.1}$$

Here, $D(x,t)$ means the local disturbance/displacement for the particle at x, at time t. For a wave on a string (like guitar string), D is the displacement *perpendicular* to the line defined by the string line in equilibrium. For sound wave propagating through any medium, D can be taken as the average displacement of local atoms[1]

[1] In the case of longitudinal sound wave, the difference of local displacements at adjacent points is directly proportional to the local pressure or the local density, which also forms the same wave form, but with a phase shift, as that of $D(x)$.

Note that above, we assumed $A \geq 0$ and $\omega \geq 0$, without loss of generality. The angular frequency (ω) is always positive or zero. The amplitude A can be taken to be non-negative by taking $\phi' \equiv \pi + \phi$ as the new initial phase if $A < 0$, since $-\sin(z+\phi) = \sin(z+\phi+\pi) = \sin(z+\phi')$ where $z \equiv kx - \omega t$.

We do *not* need to make any assumption about the sign of k, but we shall do so as a general rule, in *this* course: $k > 0$. This not only make an assumption of the sign, but it also rules out $k = 0$. This is not so much a restriction, as we shall see shortly.

Since D is a function of two variables, it is somewhat complicated.

For a fixed x value, D represents a sine function of time. Thus, each particle at each value of x goes through a simple harmonic motion! This is why the above wave is alternatively called a **harmonic wave**. The relation

$$\omega = \frac{2\pi}{T} = 2\pi f \qquad \text{angular frequency, period, frequency} \qquad (2.8\text{–}10)$$

remains valid without any need for modification.

Note that for a fixed t value, $D(x,t)$ is a sine function of x. Its periodicity in x is called **wavelength**:

$$\lambda = \frac{2\pi}{k} \qquad \textbf{wavelength (lambda)} \qquad (5.2)$$

$$k = \frac{2\pi}{\lambda} \qquad \textbf{wave number} \qquad (5.3)$$

The above sinusoidal wave is a right-moving harmonic wave. Why? To know how the above wave form moves, it is sufficient to pick one point on the wave form. We can do so by taking

$$kx - \omega t + \phi = constant \qquad (5.4)$$

where *constant* can be, e.g., $\pi/2$ (a top point of a sine wave), 0 (a mid point) or $3\pi/2$ (a bottom point). Whatever this particular point of the wave form we pick, the above equation describes how its x and t coordinates change. By taking the differential on both sides, we can figure out how our chosen point moves

$$k dx - \omega dt = 0 \qquad (5.5)$$

$$\therefore\ v \equiv \frac{dx}{dt} = \frac{\omega}{k} \qquad \text{wave velocity} \qquad (5.6)$$

The following properties follow by the same token

$$D(x,t) = A\sin(kx - \omega t + \phi) \qquad \text{right moving harmonic wave} \qquad (5.7)$$

$$D(x,t) = A\sin(kx + \omega t + \phi) \qquad \text{left moving harmonic wave} \qquad (5.8)$$

$$D(x,t) = g(x - vt) \qquad \text{right } (v > 0) \text{ or left } (v < 0) \text{ moving general wave} \qquad (5.9)$$

Note that a general wave can also be written as $g(kx \pm \omega t)$ (cf. reading quiz 3.5). However, in that form, k can be interpreted as wave number *only if* g is sinusoidal (or periodic, at the very least).

Wave vector

A much more general way of viewing k is to view it as a **wave vector**, not as a wave number. For a one dimensional wave, this can be accomplished by removing the restriction on the sign of k. For waves in higher, or general, dimensions, we can write

$$D(\vec{x},t) = A\sin(\vec{k}\cdot\vec{x} - \omega t + \phi) \tag{5.10}$$

$$\lambda = 2\pi/|\vec{k}| \qquad |\vec{k}| \text{ is the wave number} \tag{5.11}$$

The velocity of this wave is given by

$$v = \frac{\omega}{|\vec{k}|}\hat{k} \tag{5.12}$$

where $\hat{k}$ is the unit length vector given by $\vec{k}/|\vec{k}|$, i.e., the unit vector in the direction of $\vec{k}$. For one dimensional wave, this means that we can write

$$D(x,t) = A\sin(kx - \omega t + \phi) \tag{5.13}$$

$$v = \frac{\omega}{k} \tag{5.14}$$

where v can be positive or negative depending on the sign of k.

What happens if $k = 0$? This corresponds to an infinite wave length. As even the size of the Universe if not infinite, presumably, the $k = 0$ case need not be considered. Having said this, note that the $k = 0$ case is quite often discussed in physics. What is really meant is the case when λ is really large, much larger than all length scales of a problem.

5.2 Velocities

In the above we have already identified ω/k (or $\omega\,\hat{k}/|\vec{k}|$) as the velocity of the wave. This makes sense, since it means

$$v = \frac{\omega}{k} = \frac{\lambda}{T} \qquad \textbf{wave velocity} \tag{5.15}$$

where the two completely analogous relations, $\omega = 2\pi/T$ (Eq. 2.10; T represents the periodicity in time) and $k = 2\pi/\lambda$ (Eq. 5.3; λ represents the periodicity in space), are used.

Here are some more words for advanced students.

Phase velocity, Group velocity

Two different wave velocities are to be differentiated.

$$v_p = \frac{\omega}{k} \qquad \textbf{phase velocity} \tag{5.16}$$

$$v_g = \frac{d\omega}{dk} \qquad \textbf{group velocity} \tag{5.17}$$

The phase velocity is what we derived above. If you look at the derivation carefully, you will see that it is the velocity of a *single harmonic wave* with a definite frequency value. That is, **the phase velocity describes the wave velocity of a perfectly monochromatic wave**. Alas, such a wave does not exist. A **wave packet** is a more realistic form of wave. Of particular interest is a *nearly monochromatic wave*: a wave packet with a very small $\Delta\omega$, the width in ω. **A nearly monochromatic wave propagates at the group velocity, not at the phase velocity**. For this reason, generally, the group velocity is the more physical quantity!

Note that v_p and v_g differ from each other only if the medium is **dispersive**. That is, if the phase velocity is dependent on k so that ω is not a linear function of k. The functional relation $\omega = \omega(k)$ is, in general, referred to as the **dispersion relation**. For the most part, in this course, we will deal with non-dispersive cases only, i.e. $v_p = \omega/k = \text{constant} = v_g$. However, we will encounter some important dispersion physics, such as rainbow or prism. Indeed, in general, one should not assume that $\omega/k = \text{constant}$.

5.3 Transverse and longitudinal waves

If the displacement $D(x, t)$ is perpendicular to the direction of the wave propagation, then it is a **transverse** wave. If the displacement $D(x, t)$ is parallel to the direction of the wave propagation, then it is a **longitudinal** wave.

The sound wave propagating in gas or liquid is a longitudinal wave. The string wave (Figure T15-1). The same medium can support both types of waves. For instance, the sound wave propagating in a solid can be longitudinal or transverse. So is a earthquake – a seismic wave. A slinky can support a transverse wave as well as a longitudinal wave.

In the above definition, the precise meaning of the "direction of the wave propagation" is the "direction of the wave vector."