Notes for Lecture 2

Simple harmonic oscillation

2.1 Hooke's law and SHO

The motion that results from a Hooke's law force, and only that force, is called the simple harmonic oscillation (SHO).

Not any oscillation is a SHO.

Just because there is an oscillation does not mean it is a SHO! Only an oscillation caused by the Hooke's law force alone is a SHO.

What this means is that the solution to Newton's second law equation of motion (EOM)

$$m\ddot{x} = -kx \qquad \text{SHO, EOM} \tag{2.1}$$

defines the SHO, or the simple harmonic motion (SHM).

The **general solution** to the above EOM is given by any one of the following three forms[1].

$$x(t) = A\cos(\omega t + \phi) \qquad A \geq 0 \tag{2.2}$$

$$x(t) = B\sin(\omega t + \phi) \qquad B \geq 0 \tag{2.3}$$

$$x(t) = C\cos(\omega t) + D\sin(\omega t) \tag{2.4}$$

where A, B, C, D, ϕ are constants, independent of k and m, and

$$\omega = \sqrt{\frac{k}{m}} \qquad \text{angular frequency} \tag{2.5}$$

The proof of the equivalence of the above forms is left for your exercise. The following information will be quite helpful for your proof.

Trigonometry

$$\sin(a \pm b) = \sin a \cos b \pm \cos a \sin b \tag{2.6}$$

$$\cos(a \pm b) = \cos a \cos b \mp \sin a \sin b \tag{2.7}$$

That each of Eqs. 2.2,2.3,2.4 is a solution for the EOM, Eq. 2.1, can be verified by direct substitution. This verification is also left for your exercise.

Two for second derivative

The fact that there are two constants (A, ϕ or B, ϕ or C, D, respectively) for each of the above general solutions comes from the fact that the EOM, Eq. 2.1, is a second order differential equation, i.e. the highest derivative is the second order derivative. This fact remains true for any force, not just for Hooke's law force. The proof of this fact is beyond the scope of this course. You will learn the proof in an advanced course. For example, see pages 6,7,8 of this pdf file, for a preview.

[1] These three forms do not exhaust all possible forms.

Eq. 2.2 is a quite common form to use, while not the only form to use. For pedagogy, we will benefit from sticking with it, and so we will do so from now on. Eqs. 2.3,2.4 will be rarely used, if ever.

Phase, amplitude, frequency, period

The quantity $\omega t + \phi$ is called **phase**. ϕ is the **initial phase**, since it is the phase at $t = 0$. A in Eq. 2.2 is called the **amplitude**. By convention, both A and ω are taken to be non-negative.

$$T = \frac{2\pi}{\omega} \quad \textbf{period}\text{; unit = sec} \tag{2.8}$$

$$f = \nu = \frac{1}{T} = \frac{\omega}{2\pi} \quad \textbf{linear frequency}\text{; unit = 1/s = Hz} \tag{2.9}$$

$$\omega = 2\pi f = \frac{2\pi}{T} \quad \textbf{angular frequency}\text{; unit = Hz or rad/s} \tag{2.10}$$

Those are Greek!

ω is omega, not double-u. ν is nu, not v.

2.2 SHO

Let us summarize the essence of a SHO.

$$\ddot{x} = -\omega^2 x \quad \text{SHO, EOM} \tag{2.11}$$

$$x = A\cos(\omega t + \phi) \quad \text{SHO, general solution} \tag{2.2}$$

Here, the EOM is written in a simper way, using Eq. 2.5. The new form is more fundamental.

ω is also referred to as **natural (angular) frequency**. Note the fundamental nature of ω, since that is the only parameter that enters the SHO EOM, as written here in a simpler and more fundamental way than Eq. 2.1.

2.3 Velocity, acceleration

Now that we have obtained the general solution for x, we can proceed to calculate v and a.

$$v = \dot{x} = -A\omega \sin(\omega t + \phi) \tag{2.12}$$

$$a = \dot{v} = -A\omega^2 \cos(\omega t + \phi) \tag{2.13}$$

We see that when $|x|$ vanishes, $|v|$ maximizes, and vice versa. On the other hand, note that $a = -\omega^2 x$ (as expected from the EOM), and so $|a|$ and $|x|$ go up and down at the same time.

It is instructive to calculate the maximum *speed* v_{max} and the maximum *magnitude* of acceleration a_{max}.

$$v_{max} \equiv \max(|v|) = A\omega \tag{2.14}$$

$$a_{max} \equiv \max(|a|) = A\omega^2 \tag{2.15}$$