Notes for Lecture 20

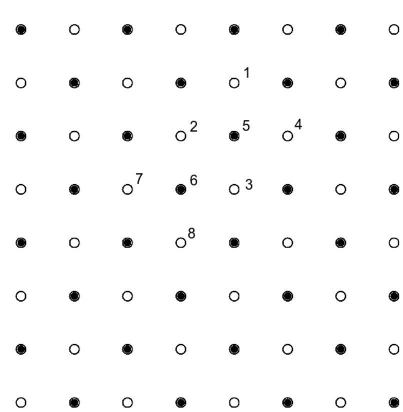
Renormalization group, cont.

In the previous lecture note, we introduced the renormalization group calculation using the 1D Ising model. Many useful concepts were introduced, there. Here, we look at the case of the 2D Ising model. Using this model, we can finally address the critical phenomena within the RG theory.

20.1 RG, 2D Ising model

Here, we will follow the renormalization treatment of the Ising model, as discussed in Wilson, Rev. Mod. Phys. 47, 773 (1975). This article is worth reading for any one interested in the RG theory.

Consider the following diagram, showing a square lattice of spins in 2D.



We will integrate out spins labelled by filled dots, while leaving those labelled by empty dots intact. Then, we will get another square lattice of spins, now 45 degrees rotated, and twice as sparse. Clearly, we can keep applying this **block spin** procedure. This is one way to coarse-grain spins, and would be the basis for setting up the RG transformation in this section.

The two dimensional Ising model has the partition function

$$\begin{aligned} X &= \sum_{\sigma_1,\sigma_2,\dots} \{\exp\left[K\sigma_5(\sigma_1+\sigma_2+\sigma_3+\sigma_4)\right] \\ &\qquad \times \exp\left[K\sigma_6(\sigma_2+\sigma_3+\sigma_7+\sigma_8)\right]\dots\} \end{aligned} \tag{20.1}$$

where the spins are labeled as shown in the following diagram.

We like to integrate out σ_5 and σ_6, and obtain an RG equation that implies a possible scaling form. By doing so, we are halving the number of spins, as we did in the previous exercise of the one dimensional case.

Effecting the sum over σ_5 and σ_6, we get

$$\begin{aligned} X &= \sum_{..,\sigma_4,\sigma_6,\dots} \{[\exp(K(\sigma_1+\sigma_2+\sigma_3+\sigma_4))+\exp(-K(\sigma_1+\sigma_2+\sigma_3+\sigma_4))] \\ &\qquad \times[\exp(K(\sigma_2+\sigma_3+\sigma_7+\sigma_8))+\exp(-K(\sigma_2+\sigma_3+\sigma_7+\sigma_8))]\dots\} \end{aligned} \tag{20.2}$$

where the sum is over the remaining spins, half as many as the initial spins.

We may hope that some sort of transformation is possible to convert this form to the original form. Let us ask if this would be possible:

$$\begin{aligned} &\exp(K(\sigma_1+\sigma_2+\sigma_3+\sigma_4))+\exp(-K(\sigma_1+\sigma_2+\sigma_3+\sigma_4)) \\ &\qquad \stackrel{?}{=} f(K)\exp(K'(\sigma_1\sigma_2+\sigma_2\sigma_3+\sigma_3\sigma_4+\sigma_4\sigma_1)) \end{aligned}$$

Of the total of 16 cases, only four cases of spin combinations are distinct.

1. All spins have the same sign:

$$2\cosh(4K) = f(K)\exp(4K')$$

2. One spin has a different sign than the other three:

$$2\cosh(2K) = f(K)$$

3. Two spins (σ_1,σ_2) have the same sign and the other two spins have the opposite sign:

$$2 = f(K)$$

4. Two spins (σ_1, σ_3) have the same sign and the other two spins have the opposite sign:
$$2 = f(K)\exp(-4K')$$

Thus, we have two conditions too many, and solutions for $f(K)$ and K' do not exist. The simplest form that works with the symmetry of the lattice and the number of distinct cases is

$$\begin{aligned}\exp(K(\sigma_1+\sigma_2+\sigma_3+\sigma_4)) + \exp(-K(\sigma_1+\sigma_2+\sigma_3+\sigma_4))\\ = f(K)\exp\Big[\frac{1}{2}K'(\sigma_1\sigma_2+\sigma_2\sigma_3+\sigma_3\sigma_4+\sigma_4\sigma_1)\\ +L'(\sigma_1\sigma_3+\sigma_2\sigma_4)+M'\sigma_1\sigma_2\sigma_3\sigma_4\Big].\end{aligned} \tag{20.3}$$

Here, $K'/2$ is used, as these nearest neighbor couplings occur exactly twice in the sum of Eq. 20.2. Now, we have exactly four cases.

1. All spins have the same sign:
$$2\cosh(4K) = f(K)\exp(2K'+2L'+M').$$

2. One spin has a different sign than the other three:
$$2\cosh(2K) = f(K)\exp(-M').$$

3. Two spins (σ_1, σ_2) have the same sign and the other two spins have the opposite sign:
$$2 = f(K)\exp(-2L'+M').$$

4. Two spins (σ_1, σ_3) have the same sign and the other two spins have the opposite sign:
$$2 = f(K)\exp(-2K'+2L'+M').$$

The solutions are

$$e^{4K'} = \cosh(4K), \quad \text{divide 1 by 4} \tag{20.4}$$
$$e^{8L'} = e^{4K'} = \cosh(4K), \quad \text{divide 4 by 3} \tag{20.5}$$
$$e^{2M'} = \frac{e^{2L'}}{\cosh(2K)}, \quad \text{divide 3 by 2} \tag{20.6}$$
$$f(K) = 2\cosh(2K)e^{M'}. \quad \text{from 2} \tag{20.7}$$

Thus, we have

$$X(K,N) = [f(K)]^{N/2} \sum_{N/2 \text{ spins}} \exp\Big(K' \sum_{\text{nn}} \sigma_i\sigma_j + L' \sum_{\text{nnn}} \sigma_i\sigma_j + M' \sum_{\text{square}} \sigma_i\sigma_j\sigma_k\sigma_l\Big) \quad (20.8)$$

where "nn" means nearest neighbors, and "nnn" means the next nearest neighbors (diagonal pairs). Notice how different this looks from the one dimensional case (Eqs. 19.5 and 19.10), where the nearest neighbor interaction went in and only the nearest neighbor term came out for the renormalized Hamiltonian. Here, we are forced to consider different quadratic terms than neighbor neighbor terms, but also quartic terms. Now, clearly, we have a problem. What happens next? If we use all these terms as input, and go to the next loop, will we get only terms of these shapes back, or will we get even higher order terms? The answer is clearly the latter.

If one were doing a professional calculation, one must keep as many higher order terms as possible, in order to produce as accurate a result as one can get. While this is clearly a non-trivial task, it is not a prohibitively difficult task either for this (rather simple) Ising model. Here, our goal is rather pedagogical, and so we will proceed by making drastic looking approximations.

First, let us ignore L' and M' completely, and see what happens. If we make this dramatic approximation, then the RG equations are given by

$$e^{4K'} = \cosh(4K), \quad (20.9)$$

$$f(K) = 2\cosh(2K). \quad (20.10)$$

This has a similar structure as the RG equations as in the 1D case (Eqs. 19.8, 19.9). There is no finite temperature phase transition within this approximation. This is an incorrect result.

As a second attempt at a better result, we shall now include K and L. Initially, $L = 0$, but as the RG loops are repeated L will grow. Then, we have the RG equations:

$$e^{4K'} = \cosh(4K),$$

$$e^{8L'} = \cosh(4K).$$

These equations are not really complete, since it applies only to the initial step starting from $L = 0$. Following Wilson, we shall now assume that K and L are small, and expand the above equations to leading orders.

$$K' = 2K^2, \quad L' = K^2.$$

What if L were non-zero, but small, to start with? Clearly to linear order in L, it will contribute to K', since the nnn interaction becomes the nn interaction after one block spin operation. Within this approximation, then, we have

$$K' = 2K^2 + L, \tag{20.11}$$
$$L' = K^2. \tag{20.12}$$

We shall limit ourselves to this approximation. These RG equations *do* have a fixed point that signify a finite transition temperature (see below for the estimate of K_c for the 2D Ising model):

$$K^* = \frac{1}{3}, \quad L^* = \frac{1}{9}. \tag{20.13}$$

Around this fixed point, we can clearly linearize the above RG equations. Using $k_1 \equiv K - K^*$ and $k_2 \equiv L - L^*$, and keeping linear terms in k_1 and k_2 only, we get

$${k_1}' = \frac{4}{3}k_1 + k_2, \tag{20.14}$$
$${k_2}' = \frac{2}{3}k_1, \tag{20.15}$$

or in the matrix notation

$$\begin{pmatrix} {k_1}' \\ {k_2}' \end{pmatrix} = \begin{pmatrix} \frac{4}{3} & 1 \\ \frac{2}{3} & 0 \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} \equiv \overleftrightarrow{A} \begin{pmatrix} k_1 \\ k_2 \end{pmatrix}. \tag{20.16}$$

We can diagonalize the matrix $\overleftrightarrow{A}$. Its eigenvalues are

$$\lambda_1 = \frac{2+\sqrt{10}}{3}, \quad \lambda_2 = \frac{2-\sqrt{10}}{3}, \tag{20.17}$$

with the corresponding eigenvectors,

$$\vec{e}_1 \propto \begin{pmatrix} 2+\sqrt{10} \\ 2 \end{pmatrix}, \quad \vec{e}_2 \propto \begin{pmatrix} 2-\sqrt{10} \\ 2 \end{pmatrix}. \tag{20.18}$$

Now, clearly any vector can be expressed as a linear combination of $\vec{e}_1$ and $\vec{e}_2$.

$$\begin{pmatrix} k_1 \\ k_2 \end{pmatrix} \equiv u_1\vec{e}_1 + u_2\vec{e}_2. \tag{20.19}$$

This defines **the scaling fields**, u_1 and u_2, which are given by

$$u_1 \propto 2k_1 + (\sqrt{10} - 2)k_2, \quad u_2 \propto -2k_1 + (\sqrt{10} + 2)k_2). \tag{20.20}$$

The field u_1 has an eigenvalue, whose magnitude is greater than 1. If we start from a certain small value $u_{1,0}$, then after n loops, we will get

$$u_{1,n} = \lambda_1^n u_{1,0}. \tag{20.21}$$

That is, $|u_{1,n}|$ grows exponentially from $|u_{1,0}|$. Such a variable is called **a relevant variable**. It is definitely relevant for describing critical phenomena. Now, for the field u_2, we see that $|\lambda_2| < 1$. Therefore, if we start from a certain small value $u_{2,0}$, then after n RG loops, we get

$$u_{2,n} = \lambda_2^n u_{2,0}. \tag{20.22}$$

And, $|u_{2,n}|$ will be exponentially small compared to $|u_{2,0}|$. Such a variable is called **an irrelevant variable**. Such a variable is usually irrelevant for the description of critical phenomena[1].

If we confine our values of k_1 and k_2 such that $u_1 = 0$, then we will be following an RG flow that always converges to the fixed point, (K^*, L^*), that we identified above. Such a geometry is generally referred to as **the critical surface**. In the current case, it is better called **the critical curve**. Points on a critical surface describe physical situations that are different microscopically. However, they all share the same critical exponents—they belong in the same **universality class**. Requiring that $u_1 = 0$, we get

$$k_2 = \frac{2}{2 - \sqrt{10}} k_1 \approx -1.72\, k_1. \tag{20.23}$$

So, the RG flow on this line flows towards $(K^*, L^*) = (\frac{1}{3}, \frac{1}{9})$, while any point near this line may follow stay near the line for a while before rapidly divering from it and converging to a trivial fixed point, $(K, L) = (0, 0)$ or $(K, L) = (\infty, \infty)$[2]. By finding the intersection of this critical curve with the $L = 0$ axis, we get a critical value, K_c, of our original 2D Ising model. $-1.72(K_c - K^*) + L* = 0$.

$$K_c \approx \frac{1}{3} + \frac{1}{9}\frac{1}{1.72} \approx 0.40. \tag{20.24}$$

The exact value of $K_c = J/(k_B T_c) \approx 1/2.269 = 0.4407$ (Eq. 18.22). Even with our crude RG calculation, we got pretty close to the exact value! In comparison, the mean field theory gives $k_B T_c = 4J$ ($z = 4$ for the square lattice), and so, $K_c = 0.25$. In addition, critical exponents such as ν can be derived. For this, the following consideration is necessary. Clearly, as we define the effective Hamiltonian (Eq. 19.17), we have

[1] But, see pages 557, 558 of Pathria and Beale, for when such an irrelevant variable becomes "dangerous" and contribute importantly. An irrelevant variable *can* affect the scaling behavior.

[2] Clearly, the second fixed point is not consistent with the assumptions that K and L be small. However, they *are* fixed points from physical arguments.

in mind that we merely plug in renormalized parameters of the Hamiltonian. This means that after we perform the block spin operation, we shrink the crystal such that the lattice constant is the same as before[3]. In other words, the length r, measured in unit of lattice constant, scales as

$$r' = \frac{1}{b}r, \tag{20.25}$$

where b is the linear dimension of the block spin (2 for the 1D Ising model RG considered in the last lecture; $\sqrt{2}$ for the current RG). And, so does the correlation length

$$\xi' = \frac{1}{b}\xi. \tag{20.26}$$

So, starting from ξ_0, after n RG loops, we have

$$\xi_n = b^{-n}\xi_0. \tag{20.27}$$

The singular part of the Gibbs free energy scales as

$$g' = b^d g \tag{20.28}$$

which simply follows from the fact that

$$N' = b^{-d}N \tag{20.29}$$

and g should scale as G/N, where G is the fixed total free energy. Thus,

$$g_n = b^{nd}g_0. \tag{20.30}$$

Consider a relevant variable, u, with the eigenvalue λ. As we change from u_0 to u_n, using n RG loops, we get

$$\xi_0 = \xi(u_0) = b^n\xi_n = b^n\xi(\lambda^n u_0), \tag{20.31}$$

and

$$g(u_0) = b^{-nd}g(\lambda^n u_0). \tag{20.32}$$

Identifying $u \propto t$, where $t = \frac{T}{T_c} - 1$, we get, by applying the definition $\xi(u) \propto u^{-\nu}$ (Eq. 17.57),

$$u_0^{-\nu} = b^n\left(\lambda^n u_0\right)^{-\nu}. \tag{20.33}$$

[3]This shrinking was implicit in the discussion of Section 19.1.

Taking the log, we get

$$\nu = \frac{\log b}{\log \lambda}. \tag{20.34}$$

From the above RG calculation, we get $\lambda_1 \approx 1.72$, and $b = \sqrt{2}$, and so $\nu = 0.64$. The correct value is 1 (Eq. 18.23).

In the above expression for ν, it is clear that ν must be independent of b. Writing $\lambda = b^y$, we get $y = 1/\nu$. Clearly, $\lambda = b^y$ form is valid for any variable in the linear regime, since $\lambda_1\lambda_2 = b_1^y b_2^y = \lambda_{tot} = (b_1 b_2)^y$, if the block spin transformation is taken first with b_1 and then second with b_2.

20.2 General discussion

Now, consider a more general problem with a finite magnetic field. Above, we have seen that there is a relevant variable. When k_1 is expressed as a function of temperature, it is $\propto t$, where t is the reduced temperature. On general grounds[4], we also expect that there to be a relevant variable $\propto h$, for small h. So, the singular part of the free energy per spin reads

$$g(t, h, ...) = b^{-nd} g(b^{ny_t} t, b^{ny_h} h, ...) \tag{20.35}$$

where the eigenvalue for t is given as b^{y_t} and the eigenvalue for h is given as b^{y_h}. Let us fix a small $t_0 = b^{ny_t} t$. Then $b = \left(\frac{t}{t_0}\right)^{-1/(ny_t)}$.

$$g(t, h, ...) = \left(\frac{|t|}{t_0}\right)^{-d/y_t} g\left(t_0, \left(\frac{|t|}{t_0}\right)^{-\Delta} h, ...\right), \tag{20.36}$$

where

$$\Delta \equiv \frac{y_h}{y_t}. \tag{20.37}$$

As the left hand side is independent of t_0, so should be the right hand side, and so we get

$$g(t, h, ...) = |t|^{-d\nu} \phi(h|t|^{-\Delta}, ...) \tag{20.38}$$

[4]Applying H field decreases the correlation, and moves the system away from the critical point. So, the non-trivial fixed point occurs at $h = 0$ only.

where ϕ is a scaling function. As in Pathria and Beale, and other standard books, it can be shown that the following relationships for the critical exponents follow from this.

$$\alpha = 2 - d\nu, \quad \beta = 2 - \alpha - \Delta, \quad \gamma = -(2 - \alpha - 2\Delta), \quad \delta = \Delta/\beta. \tag{20.39}$$

Lastly, from the consideration of the spin correlation function, one can deduce (page 543 of Pathria and Beale):

$$\eta = d + 2 - 2y_h. \tag{20.40}$$

This can be re-written as

$$\Delta = \frac{\nu}{2}(d + 2 - \eta). \tag{20.41}$$

Using this, the above four **hyperscaling relations** can be rewritten as

$$\alpha = 2 - d\nu, \quad \beta = -\nu(2 - d - \eta)/2, \quad \gamma = \nu(2 - \eta), \quad \beta\delta = \nu(2 + d - \eta)/2. \tag{20.42}$$

So, out of six critical exponents, $\alpha, \beta, \gamma, \delta, \eta, \nu$, only two are independent, and they determine all other exponents. In the last equation, it is clear that knowing ν and η (and d), one can determine all other exponents[5].

[5] However, a warning. Note that, for the Ising model in dimensions greater than or equal to 4, the mean field theory is valid (Section 17.7). This means that critical exponents must be independent of d in these cases, while the above hyperscaling relations are not! It turns out that for the Ising model with $d \geq 4$, the above hyperscaling relations must be changed ($d \to 4$). This problem is related to the (unjustified) ignoring of "dangerous" irrelevant variable in the derivation of the hyperscaling relation. See footnote 1.