Notes for Lecture 19

Renormalization group

In the previous lecture note, we have addressed methods that can give exact solutions. However, one must recognize that exact solutions are often not as good as they sound. Numerical solutions are limited by the capacity of computers. Exact analytical solutions are exceptions rather than rules.

This is why often approximate methods are the best general things that we can rely on. In the context of the statistical mechanics or the many body theory, the renormalization group theory is quite central for this reason. It is a theory whose main objective is to find "fixed points," or low energy effective Hamiltonians, and behaviors around fixed points. The question that we ask in the renormalization group theory is: **How does a given Hamiltonian approach a fixed point? This question is equivalent to: What happens to the partition function as we do the integration, one step at a time?** If each step (a loop in the computational sense) of the calculation can be carried out exactly, then the renormalization group calculation will give an exact result. In this lecture, we will demonstrate this. Even if each step of the calculation is done approximately, the result is easily better than the mean field theory, and can be improved systematically to a better result. For this reason, the range of application of the renormalization group theory has been proven to be very broad, much broader than exact solutions.

19.1 Scaling and renormalization

The "group" in the expression "renormalization group" refers to the mathematical group. And, it is a misnomer, mathematically speaking. However, the motivation is clear enough. The group theory forms a very important backbone of physics.

This is because the group theory is the mathematical language of the symmetry and conservation principles of physics. Here, we are also talking about a sort of symmetry, indeed.

What symmetry are we talking about here? It is the kind of symmetry that we call **scale invariance**.

To start the discussion on the scale invariance, we note that a power law function is a scale-invariant function. What do we mean by this? Intuitively speaking, it means the following. In this discussion we consider a function $f(x)$, where x represents length, and we are interested in the length scale associated with the coordinate x. Take a function such as a Gaussian function. It has a bell shape with width $\Delta x \sim \sigma$. Such a function is *not* scale invariant. Rather it has an intrinsic length scale, σ. What do we mean by this? Imagine that we have a mountain that looks like a Gaussian function. Suppose that you are climbing on that mountain. So, you are very close to the mountain, obviously. So very close to the mountain, the shape of the mountain does not look like a Gaussian function at all. You could not know the whole shape of the mountain, if you are very close to it. The shape of the mountain simply looks like a linear slope that challenges your climbing ability and physical strength. If you are reasonably far away from the mountain, then you will recognize that the mountain looks like a Gaussian shape. If you now go very far away, e.g., if you are flying in an airplane that keeps going away from the mountain, then eventually the mountain looks like a dot. As you can see, the shape of a mountain is not scale invariant. Most functions are not scale invariant.

However, please try to convince yourself that if a mountain slope is in the shape of a power law, $f(x) = x^{-p}$, then, the shape of the slope will be the same as you look at it at every scale! Mathematically, this is noted as: $f(\lambda x) = f(x)\lambda^{-p}$. This is the definition of **scale invariance** of a function. Now, changing the notation slightly, let us assume

$$\Gamma(\lambda r) = \frac{\Gamma(r)}{\lambda^p}. \tag{19.1}$$

It is left for your exercise to prove that such a function is necessarily a power law function, $\Gamma(r) = Ar^{-p}$, with a constant multiplicative factor A to be determined by additional constraint.

Let us note that we have seen this type of function! It is **the correlation function** at $T = T_c$, studied in lecture 17 (Eqs. 17.56, 17.58). That is, the correlation function becomes scale invariant at $T = T_c$: with $p = d + \eta - 2$.

Note that the scale invariance discussed above is reminiscent of self-similarity, as that for a fractal geometry.

Now, let us ask—what kind of state is a critical state, then? To answer this important question, it is convenient to focus our Ising model, as we have done in the previous lectures, and fix σ_1 to be $+1$ for qualitative thinking. As shown in LN 17 (Eq. 17.19), the rest of the spin configuration will be arranged to show the spin correlation function $\Gamma(r)$, on average. That the correlation length is infinite means that the positive σ patch nucleating from σ_1 extends throughout the sample. However, such a patch necessarily has to be filamentary as opposed to blob-like, on a macroscopic length scale, intermingled with similarly filamentary patch of $\sigma = -1$ with an infinite correlation length. This is because, on the average, the net spin for any finite fraction of the system must be zero.

If this description calls up in your mind a picture of fractal like geometries, that is very excellent. Indeed, it is a fundamental assumption of the **the renormalization group theory** that the system at a critical point is invariant under scaling transformation. It must be noted, though, that the length scales discussed here are limited to lengths much larger than the atomic length scale. This requires us to define the description of the system in terms of **coarse grained length variables**[1]. To summarize, the system looks self-similar in all length scales after sufficient coarse-graining of the length scale. This coarse graining procedure, when defined properly in the following sections, is called **the renormalization group transformation** or **the scaling transformation**.

Before going to actual calculations, the following consideration is useful. Still holding σ_1 to be 1, consider what happens to the system as the temperature is lowered, starting from well above T_c. At the initial temperature, the temperature is so high that the exchange interaction J can be ignored: we have the Curie paramagnet behavior (Lecture 8). At this initial temperature, σ_1 cannot nucleate any magnetic patch at all. The correlation length is one lattice constant or two, i.e. completely negligible in the macroscopic sense. Now, lower the temperature. Short range patches begin to form and they fluctuate in time. We see that a small patch appear around σ_1, and it becomes longer and longer in extent. As T approaches T_c, we have this patch permeating the entire system. As remarked above, all this while, the system is not ordered, and any finite part of the sample does not have any net spin[2]. Now, what happens when T is lowered further, slightly below T_c? Now, the system forms a small (irregular) blob of spin nucleating from σ_1 in a way that the total net spin is positive. Why? It is because the fixed spin $\sigma_1 = 1$ acts like a magnetic field and tips the balance of the neighboring spins towards positive value of spin. In turn, these neighboring spins do the same to their neighboring spins. Therefore, we should expect that the

[1]The coarse grained variable was introduced in Section 17.6, without any proper definition. The proper definition is the topic of this lecture.

[2]Note that not only spatial average, but also temporal average, must be applied here. The time average corresponds to a "coarse graining in time," averaging over time scale much larger than the atomic time scale.

one fixed spin be enough to tip the balance of the symmetric potential to cause a symmetry breaking (Figure in page 10 of Lecture 16, tilted by the addition of $-mh$ with a small h). In this sense, a small magnetic field or a fixed spin like this one can be called a symmetry breaking field. When T is lowered further to zero, this blob bigger and bigger and occupies the whole system. A good alternative way to think about the condensed phase below T_c is to start from $T = 0$, rather than from T_c. At $T = 0$, all spins are aligned in one direction ($\sigma = 1$ due to σ_1). As T is raised, however, spins begin to fluctuate due to thermal fluctuation—finite size patches of disordered spins begin to appear, "eating away" some of the net magnetization. This disorder patches, or the fluctuation patches, do not extend to the entire system below T_c—they have the finite correlation length. As T_c is reached, the spin fluctuation becomes infinite in its extent, and thus it finally permeates throughout the system. When it does, the net magnetization vanishes.

So, a critical state is a scale-invariant state. And the "scaling transformation" or the "renormalization group transformation" can be considered a "symmetry operation." But, note that the scaling transformation is a one-way operation—a "zooming-out" operation or a "coarse-graining" operation. Zooming out operations, without their inverses, cannot form a mathematical group. Thus, a bit of a misnomer. Also, **the renormalization group operations are not uniquely defined**. This is different from other more clearcut symmetry operations such as translation, reflection, rotation, charge conjugation etc. These are well-defined operations. However, the renormalization group operations are, as we shall see, somewhat loosely defined. For a given system of spins, researcher A might define "zooming out" as deleting every other spin and computing the effective Hamiltonian for the rest of the spins, while researcher B might work in k-space and integrate out high k components of the Hamiltonian to define the effective Hamiltonian. In either case, it is important to note that the core process of integrating out short range physics is common. Thus, the renormalization group seeks to examine the behavior at large length scales, or, equivalently, at small energy scales. As long as this process of integrating out is correctly done, the correct physics will be captured, and the details of the actual "symmetry operation" can be defined somewhat idiosyncratically.

From the above discussion, we see that a critical state would be a fixed point of a renormalization group transformation. What if we are very close to the critical point, characterized by a small t value or a small h value? As we zoom out, we will see that initially, there is some sort of scale invariant behavior as long as our coarse-grained variable $r \ll \xi$. However, as we pass $r \sim \xi$ and continue to $r \gg \xi$, we will realize that the system definitely has a finite length scale. As we shall see later, this type of behavior can be discussed most conveniently in terms of a **renormalization group (RG) flow** in the parameter space. If we start from a true critical point, then the RG flow will go to a fixed point. If we start from a point close to a critical point, then

initially it may look like we are approaching the same critical point. However, as the coarse grained length scale surpasses ξ, the RG flow will deviate and diverge from the fixed point. This behavior is the essence of the so-called **scaling hypothesis**: sufficiently close to the critical point, ξ is the only length scale of the problem relevant, and the correlation function of the system is given as $|t|^q f(r/\xi)$. In the RG theory, such hypothesis receives a theoretical support.

To summarize, the "renormalization" in the name "renormalization group" refers to the fact that the parameters of a given physical model change from their initial values as the renormalization group transformation is applied. The way the parameters get renormalized tell us what the essential character of the low energy physics is. In particular, at a critical point, we expect that these parameters not be renormalized after sufficient steps of renormalization, as the system must be scale-invariant. So a critical point flows to a fixed point. However, the converse is not true. Just because we have a fixed point does not mean a critical phenomenon around or at that point. It can be a trivial fixed point (characterized by zero correlation length, not infinite correlation length!), as we shall see shortly.

Many of these points can be demonstrated by a simple model: the one dimensional Ising model.

19.2 RG, 1D Ising model

Here, we will use a very simple renormalization group theory on a one dimensional (1D) Ising model, as a nice "warm-up" problem to a more significant two dimensional problem. The 1D problem is a good place to start, since it is simple case to demonstrate the structure and the power of the RG theory. Not surprisingly, we can easily obtain the exact solution of the 1D Ising model, using the RG theory! However, since we are more interested in a finite temperature phase transition, this exercise is a warm-up for the 2D case.

The Hamiltonian of this problem is given by

$$\mathcal{H} = -J\sum_i \sigma_i\sigma_{i+1} - H\mu_B\sum_i \sigma_i. \tag{18.8}$$

We shall consider the case where $H = 0$ and $J > 0$. At the end of this section, we will discuss what happens if we consider other cases. The Gibbs partition function for $H = 0$ is given by

$$X(K,N) = \sum_{\{\sigma_i\}} e^{K(\sigma_1\sigma_2+\sigma_2\sigma_3+\ldots+\sigma_N\sigma_1)}, \tag{19.2}$$

$$K \equiv J\beta, \tag{19.3}$$

where the periodic boundary condition is used for N spins. Notice that, normally, the Gibbs partition function for $H = 0$ must be written as a function of T and N. Here, K replaced T as an independent variable. We can coarse-grain simply by integrating out every other spin, e.g. spin at even indices. The result is

$$X(K,N) = \sum_{\{\sigma_1,\sigma_3,\ldots\}} \left\{e^{K(\sigma_1+\sigma_3)} + e^{-K(\sigma_1+\sigma_3)}\right\} \cdot \left\{e^{K(\sigma_3+\sigma_5)} + e^{-K(\sigma_3+\sigma_5)}\right\} \cdot \ldots \tag{19.4}$$

Now, this looks quite different from the original partition function, at first sight. However, we are looking for a possible scale invariant form. So, we might hope to put this new partition function in a form that looks like

$$\sum_{\{\sigma_1,\sigma_3,\ldots\}} e^{K'(\sigma_1\sigma_3+\sigma_3\sigma_5+\ldots)}.$$

This is impossible to do, but, it is possible to achieve something like this, if we allow a multiplicative factor at the front. Namely, it will be possible to accomplish something like this if we require that

$$e^{K(\sigma_i+\sigma_{i+2})} + e^{-K(\sigma_i+\sigma_{i+2})} = f(K)e^{K'\sigma_i\sigma_{i+2}} \tag{19.5}$$

where f and K' are two unknowns to be determined. That these two quantities can be determined uniquely follows from the fact that our requirements consist of just two equations[3]

$$2\cosh(2K) = f(K)e^{K'} \qquad \text{if } \sigma_i\sigma_{i+2} > 0, \tag{19.6}$$

$$2 = f(K)e^{-K'} \qquad \text{if } \sigma_i\sigma_{i+2} < 0. \tag{19.7}$$

These have the following solutions

$$e^{2K'} = \cosh(2K), \tag{19.8}$$

$$f(K) = 2\sqrt{\cosh(2K)}. \tag{19.9}$$

And the partition function can be written as

$$X(K,N) = 2^{N/2}\cosh(2K)^{N/4}\, X(K',N'), \tag{19.10}$$

where $N' \equiv \frac{N}{2}$.

These last three equations constitute **the renormalization group (RG) equations**. If we take the log of the last equation and define

$$\log X(K,N) \equiv -N\beta g(K) \qquad g \text{ is the Gibbs free energy per spin} \tag{19.11}$$

we get the following result.

[3] $f(K) = 1$ is clearly a non-solution, consistent with the discussion we just had.

The Gibbs free energy per spin (or block spin), g, transforms according to

$$-\beta g(K) = -\frac{1}{2}\beta g(K') + \frac{1}{2}\log\left(2\sqrt{\cosh(2K)}\right) \tag{19.12}$$

$$= -\frac{1}{2}\beta g(K') + \frac{1}{2}\log f(K) \tag{19.13}$$

as the system is coarse-grained by the RG transformation.

Another way to think about the RG transformation arises from the following identity:

$$X(K,N) = \mathrm{tr}\left\{e^{-\beta\mathcal{H}}\right\} = \mathrm{tr}\left\{e^{-\beta\mathcal{H}'}\right\}, \tag{19.14}$$

where $\mathcal{H}'$ is **the effective Hamiltonian** for the course-grained system. Since

$$X(K,N) = f(K)^{N/2} \sum_{\{\sigma_1,\sigma_3,\ldots\}} e^{K'(\sigma_1\sigma_3+\sigma_3\sigma_5+\ldots)} \tag{19.15}$$

we get

$$-\beta\mathcal{H}' = K'\{\sigma_1\sigma_3 + \sigma_3\sigma_5 + \ldots\} + \frac{N}{2}\log f(K) \tag{19.16}$$

$$= -\beta\mathcal{H}(J',N') + \frac{N}{2}\log f(K), \qquad J' \equiv K'/\beta \tag{19.17}$$

where $\mathcal{H}(J',N')$ is the same *form* of the Hamiltonian as Eq. 18.8 (with $H = 0$), except that it applies only to the coarse-grained spin lattice (so, N') and J is replaced by J'.

So, in this view, the effective Hamiltonian of the new coarse-grained system, $\mathcal{H}'$, is to be understood as essentially of the same form as the original Hamiltonian, $\mathcal{H}$, since all we need to do is to apply the original Hamiltonian in the coarse grained lattice with **the renormalized parameter** $J' = K'/\beta$, instead of the original parameter, $J = K/\beta$, as the first term shows, and then shift the energy by $-\frac{N}{2\beta}\log f(K)$, as indicated by the second term, which is just a number, not a quantum-mechanical operator.

While this shift term appearing in the above expressions for $\mathcal{H}'$ and g *is* important for calculating the partition function correctly, for sure, it is called **the non-singular**

term for $\mathcal{H}'$ or g (or G), and, you will see in many papers that it is *ignored* for the purpose of calculating critical behaviors around a phase transition. On the other hand, the first term appearing in the above expressions for $g(K)$ and $-\beta\mathcal{H}'$ is called **the singular term**, and must be kept along for the discussion of critical behaviors. The reason for these terminologies is very simple. The function $f(K)$ above (or the logarithm of it) is an analytic function of thermodynamic variables for any non-zero temperature, for the same reason why a true phase transition can occur in the thermodynamic limit ($N, V \to \infty$): a finite sum over an analytic function (Boltzmann factor) is an analytic function and an analytic function cannot describe a sudden turn on of an order parameter associated with a phase transition[4]. And, so any non-analytic behavior of the free energy must arise from the first term: this is the reason why it is called **the singular term**, in the sense that it is responsible for any singularities of free energy and response functions at the phase transition.

Clearly, the above equations for g and $\mathcal{H}'$ are recursion relations. For g, one starts with K and $g(K)$. Next, one obtains $f(K)$ and K' from RG equations. Then, one gets $g(K')$. Call K' a new K, and the same process repeats. For $\mathcal{H}'$, one starts with K and $\mathcal{H}(K, N)$. Using RG equations, one obtains N', K' and $f(K)$. Then, one obtains $\mathcal{H}'$. Call K' a new K, and N' a new N, and then repeat the same process for the singular part of the Hamiltonian.

Here is an intuitive way to think about the renormalization process, in general. As we run renormalization loops, we will see that the parameters of the Hamiltonian follow a trajectory, which we call **an RG flow**. Such a flow reveals the large length-scale, or the macroscopic, nature of the system. All short length-scale, or the large energy-scale, physics is *integrated out* in the shift term discussed above, cumulatively, as the renormalization group transformation loops are applied, and **the true macroscopic nature of the system is revealed in the effective Hamiltonian**, as only the singular part determines which phase (ordered, disordered, or critical phase) the system belongs.

Now, let us get back to what can we learn specifically about the 1D Ising model. Note that the first RG equation, Eq. 19.8, can be rewritten as

$$e^{2K'} + e^{2K'} = e^{2K} + e^{-2K}.$$

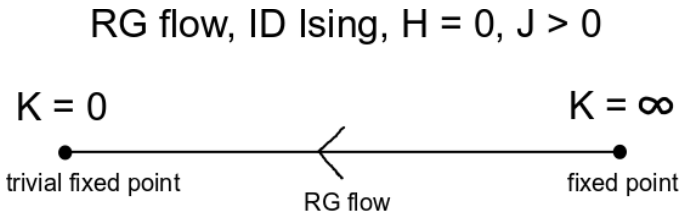
As we are considering the ferromagnetic case $K \geq 0$ only, we note that $K' < K$ if $K > 0$, and $K' = K$ if $K = 0$. Thus, $K = 0$ is **a fixed point** for this RG transformation. Another fixed point is $K = \infty$. K' is renormalized with respect to K by application of the RG equation. In order to examine the low energy behavior of a given system,

[4]Note, however, that despite what we have been discussing so far, there are phase transitions that do *not* involve an order parameter. A Kosterlitz-Thouless transition is an example. How is such a transition defined? The non-analyticity of thermodynamic quantities, among others!

the above RG equations must be applied repeatedly in loops, by setting $K = K'$ at the end (or the beginning) of each loop. Thus, in the RG equation, K is a **running coupling constant**. As the RG equation is repeatedly applied from a finite value of K, K goes to 0. In the RG language, we say that the parameter K follows the RG **flows** to the $K = 0$ fixed point.

What is the nature of the $K = 0$ fixed point? It is where the effect of the interaction is zero, either due to J being zero or T being very high. It is the disordered phase. While such a state is also scale invariant, it is not scale invariant for an interesting reason. **It is a trivial fixed point, since the correlation length is zero, instead of infinite.** Note that for the scale invariance to hold, the correlation length can be either infinite or zero.

The other fixed point is $K = \infty$, which represents the $T = 0$ phase. This is a completely ordered phase. At $T = 0$, the system is fully magnetized. It is clear why such a state is scale invariant. However, this scale invariance is not as interesting as the scale invariance at a finite critical temperature. However, this is not the only character of this fixed point. **It is also a non-trivial fixed point, since $\xi \to \infty$ at this point**[5]**.** So, this fixed point can be said to have a dual character. Recall, from lecture 17, that $\xi = \infty$ means the following: for a sample in thermal equilibrium, flip one spin and hold it (e.g., by planting a magnetic impurity), and the system will generate a static pattern of spin density going all the way to the edge of the sample. The RG flow shows what happens when the system is viewed in longer and longer length scales. Without a long range order, the system eventually looks disordered at sufficiently large length scale. The fact that in the current example the RG flows to the $K = 0$ fixed point, starting from any finite value of K, means that there is no long range order at any finite temperature. This is consistent with the exact solution (Section 18.1.1).



[5] $\xi = [\log \coth K]^{-1}$. So, the correlation length diverges exponentially, not as a power law, as $T \to 0$. The derivation of the formula for ξ is left for your work. Here are some guidelines. First, the correlation function (Eq. 17.6) in the disordered phase is given by $\Gamma_j = \overline{\langle \sigma_1 \sigma_j \rangle} = \overline{\langle \sigma_1 \sigma_2 \sigma_2 \sigma_3 ... \sigma_{j-1} \sigma_j \rangle}$, since $\sigma_i^2 = 1$. Then, write $-\beta H = \sum K_i \sigma_i \sigma_{i+1}$, where K_i's are taken to be independent variables that will be put to K at the end of the calculation. The partition function for this Hamiltonian can be obtained as easily as that for uniform K (steps leading to Eq. 18.6). And, $\Gamma_j = \frac{1}{X} \frac{\partial^{j-1} X}{\partial K_1 \partial K_2 ... \partial K_{j-1}}$. The result: $\Gamma_j = \tanh^j(\beta J)$.

What would have happened if there *is* a finite temperature critical point, T_c? We would have found that the corresponding *finite* value of $K_c = J/(k_B T_c)$ is a fixed point. We will see later that this happens in higher dimensions. In such a case, we can examine how the RG flow behaves near that fixed point and extract critical exponents. All these discussions can be done keeping only the singular part of the Gibbs free energy or the Hamiltonian in mind.

For the current case, we already know that such singular part does not exist, since there is no phase transition at any finite temperature (the above discussion as well as the exact solution derived in LN 18). This is quickly confirmed as follows. Consider applying Eq. 19.10 recursively, all the way down to $N' = 1$. At that point, $K' = 0$ *always*, as we found above. So, $X(K' = 0, N' = 1) = 1$. If a finite temperature phase transition existed, then we will find that we have a fixed point at a finite value of K. That would be a non-trivial fixed point, corresponding to the critical point. We will also have two trivial fixed points, one associated with the disordered phase, and the other associated with the ordered phase. If K is chosen randomly, then it will most likely correspond to an ordered phase or a disordered phase. How would we see it in the RG theory? The RG flow will take the original point to either one of the two trivial fixed points. If the two initial values of K are very close to each other, but on different sides of the critical point, then we will get completely different values of final K values. Thus, the discontinuity of the free energy.

In the next lecture, we will see that happening. For now, we are content with the problem that has a trivial fixed point and a non-trivial fixed point at $T = 0$.

One thing that is worth noting is that the RG equations that we set up above for the 1D Ising problem contains no approximation: *they are exact*. So, if we use the above recursion relations, then the free energy must come out as exact. From Eq. 19.13, and noting that $g(K' = 0, N' = 1) = 0$ (since $X(K' = 0, N' = 1) = 1$), we get

$$\begin{aligned}
-\beta g(K) &= \frac{1}{2}\left(\log f(K) + \frac{1}{2}\log f(K') + \left(\frac{1}{2}\right)^2 \log f(K'') + ...\right) \\
&= \frac{1}{2}\left(\log\left(2\cosh(2K)\right) + \frac{1}{2}\log\left(2\cosh(2K')\right) + \left(\frac{1}{2}\right)^2 ...\right) \\
&= \log 2 + \frac{1}{2}K' + \left(\frac{1}{2}\right)^2 K'' + \left(\frac{1}{2}\right)^3 K''' + ... \qquad \text{using Eq. 19.8}
\end{aligned}$$

So, in this RG calculation, evaluating the free energy comes down to evaluating the series $\frac{1}{2}K' + \left(\frac{1}{2}\right)^2 K'' + \left(\frac{1}{2}\right)^3 K''' + ...$ This series can be evaluated by noting that the recursion relation, Eq. 19.8, can be re-written, using $\cosh K = \sqrt{\frac{\cosh(2K)+1}{2}}$, as

$$\cosh K = e^{\frac{K'}{2}} \sqrt{\cosh K'}. \tag{19.18}$$

Taking the logarithm and applying this equation recursively, we get

$$\log\cosh K \;=\; \frac{1}{2}K' + \left(\frac{1}{2}\right)^2 K'' + \left(\frac{1}{2}\right)^3 K''' + ... \tag{19.19}$$

Using this result, we can easily see that the free energy is given by

$$-\beta g(K) \;=\; \log(2\cosh K) \tag{19.20}$$

in precise agreement with the exact solution, Eq. 18.7. **This demonstrates the fundamental principle that an RG calculation, if carried out without any approximation, gives the exact solution.** In a higher dimensional problem, it is generally found that it is not so easy to carry out the RG calculation exactly, due to the proliferation of higher order terms. Indeed, it is a nice peculiarity of the 1D Ising model that the RG transformation can be set up so nicely involving only the quadratic terms of σ operators (terms of the form $\sigma_i\sigma_j$). In general, fourth order terms, sixth order terms, and higher order terms spring into the effective Hamiltonian during the RG transformation. That they spring into existence is annoying, as it makes the problem difficult to handle, but is also a perfectly sensible thing to expect, since, as we discussed with the Landau theory (LN 17, pages 16 and 17), the effective Hamiltonian is the Gibbs free energy from the point of view that small length scale physics that have been integrated out by the renormalization calculation. So, for general RG problems, we do not have as easy time as we have here for the 1D Ising model. However, even when we cannot easily set up the RG equations exactly, it is quite easy to do better than the mean field theory, as we will show in the next lecture. Furthermore, with a bit more efforts to include sufficient number of terms, the calculation may be made essentially exact for a sufficiently simple model—such is the value of the RG theory!

As we have noted in this lecture, and also in the previous lecture, the free energy that we obtained above is analytic at any finite temperature. However, it is *not* analytic at $T = 0$ (or $J = \infty$), the point that doubles as the critical point (a non-trivial fixed point) and the completely ordered phase (normally, a trivial fixed point).

The discussion of the RG equation in the proximity of the non-trivial fixed point is of importance in understanding critical behaviors. For the current model, such discussion is *not* easy for two reasons. (1) In the current problem, the non-trivial fixed point is at $K = 0$. Note that K is essentially a temperature variable. To measure the temperature from a critical point, we have adopted $t = \frac{T}{T_c} - 1$ as a dimensionless variable in previous lectures. Clearly this is not a good idea if $T_c = 0$. Instead, however, we define the following small parameter,

$$x \;\equiv\; e^{-4K} \tag{19.21}$$

which is a physically meaningful parameter, since it represents the Boltzmann factor for flipping one spin on a completely ordered state. One *can* discuss the critical

behaviors with this parameter, as *the scaling variable*. (2) The second difficulty is that, as we have mentioned above, the separation of the singular part and the non-singular part of the free energy is not possible for the $T = 0$ critical point. For instance, if we take the exact solution for g, and expand it near $x = 0$, then we get

$$-\beta g \;=\; -\frac{\log x}{4}\log(1+\sqrt{x}) \;=\; -\frac{\log x}{4}\left(1+\sqrt{x}+...\right). \tag{19.22}$$

So $-\beta g$ diverges logarithmically. Blind calculations keeping track of only singular terms of the free energy in the above RG equation (Eq. 19.13) will instead give the result $-\beta g \sim x^{1/2}$, which is not really correct.

One may ask, what if $J < 0$? The RG transformation set up above is not appropriate for studying the anti-ferromagnetism. One may note that, starting from a negative value of K, the first application of the RG equation above immediately gives a positive K, turning the model into a ferromagnetic one. This does *not* mean that an anti-ferromagnetic Ising model is mapped to a ferromagnetic Ising model. It means that the RG transformation set up above is not good enough to study both the ferromagnetic Ising model and the anti-ferromagnetic Ising model. This is clear from the fact that $K = -\infty$ is *not* a fixed point if we use the above RG equation. However, no worries. Nobody said that the RG transformation is unique. In fact, we said quite the opposite (page 4). It is left for your exercise to set up an RG equation applicable to the $J < 0$ case (hint: sum over two out of three adjacent spins, not one out of two adjacent spins, at a time).

Finally, one can also ask, what if $H \neq 0$? The above is easily generalizable in this case. We can define

$$h \;=\; \frac{\mu_B H}{k_B T} \tag{19.23}$$

as a dimensionless variable[6] for H. Then, the Hamiltonian is given by

$$-\beta H \;=\; h\sum_i \sigma_i + K\sum_{\langle i,j\rangle}\sigma_i\sigma_j. \tag{19.24}$$

Accordingly, the RG equations can be derived from the following requirement

$$\left(e^{K(\sigma_1+\sigma_3)+h}+e^{-K(\sigma_1+\sigma_3)-h}\right)e^{h\frac{\sigma_1+\sigma_3}{2}} \;=\; f(K,h)e^{K'\sigma_1\sigma_3}e^{h'\frac{\sigma_1+\sigma_3}{2}} \tag{19.25}$$

or the same equation with indices $1, 3$ replaced by $i, i+2$, with $i = 1, 3, 5, 7, ...$ Again,

[6]This definition is similar to, but not quite the same as, the definition that we used in previous lectures (15-17) in that we divide by T, not by T_c of the mean field theory.

it is found that the following three RG equations can be derived exactly.

$$e^{2h'} = e^{2h}\frac{\cosh(2K+h)}{\cosh(2K-h)}, \tag{19.26}$$

$$e^{2K'} = \frac{\sqrt{\cosh(2K+h)\cosh(2K-h)}}{\cosh h}, \tag{19.27}$$

$$f(K,h) = 2\sqrt{\cosh h}\,\sqrt[4]{\cosh(2K+h)\cosh(2K-h)}. \tag{19.28}$$

Note that $K = 0$ is a fixed point for any value of h. This is a line of trivial fixed points. There is no correlation ($\xi = 0$), if $T = \infty$ or $J = 0$. There is also a non-trivial fixed point ($\xi = \infty$): $K = \infty$ and $h = 0$. The RG flow diagram is given below. Note that as before, we are implicitly using exponential variables, $u = e^{-2K}$ and $v = e^{-2h}$, for the x axis and the y axis, respectively, so that, e.g., the full range from $K = 0$ to ∞ can be represented compactly.