Due Apr. 17, Thursday.

Problem 1 (15 points) Using the Sterling's formula (Eq. 4.35) and the Taylor expansion of the logarithm, prove that the binomial distribution (Eq. 3.25) becomes the Gaussian distribution, in the limit of $Nh \to \infty$, $N(1-h) \to \infty$. Hint: Write $n = Nh + x$, and note that $|x| \ll Nh$ when Nh is large. Take the logarithm of the binomial distribution, obtain the leading order expansion for x.

Problem 2 (20 points) Consider a Lorentzian distribution function Eq. 3.55:

$$p(\omega) = \frac{1}{\pi} \frac{\Gamma}{(\omega - \omega_R)^2 + \Gamma^2} \qquad \Gamma > 0.$$

(a) By contour integration, or any other method that you see fit, prove that $\int_{-\infty}^{\infty} d\omega\, p(\omega) = 1$. Note that the mean of this distribution is not defined, strictly speaking. However, the median is well-defined, and it is ω_R. The full width at half maxima (FWHM) is also well-defined, 2Γ, while the standard deviation is not.

(b) Show that the characteristic function $\tilde{p}(t) \equiv \langle e^{-i\omega t} \rangle$ does exist, despite the fact that most moments and cumulants are ill-defined. Find it (by contour integration or some other method that you see fit).

(c) Suppose that the above distribution is an approximate description of the frequency distribution of photons from a certain atom. Furthermore, let us assume that we have N atoms, each of which can be different from one another. Then, we can consider the following sum variable

$$\omega = \omega_1 + \omega_2 + \ldots + \omega_N$$

where each ω_j ($j = 1, \ldots, N$) satisfies the following distribution separately (and thus, independently of one another)

$$p(\omega_j) = \frac{1}{\pi} \frac{\Gamma_j}{(\omega_j - \omega_{R,j})^2 + \Gamma_j^2} \qquad \Gamma_j > 0.$$

Find the probability distribution for ω, and show that it is a Lorentzian distribution for any N value, demonstrating that the central limit theorem does not apply here. What is the median value of ω? How about the FWHM?

[Note] Here, we considered ω to be an unrestricted real number. One might point out that, in an actual experiment, frequencies, such as ω_R and ω, must be non-negative, by definition. How can we invoke negative frequencies? (1) The many-body Green's function *does* have a negative frequency part, which corresponds to the time-inverted process of the positive frequency part, and the spectral weight, which is proportional to the imaginary part of the Green's function and is approximately given by a Lorentzian function in many cases, normalizes to one, only if integrated from $-\infty$ to ∞. (2) In common cases, additionally, the condition $\Gamma \ll \omega_R$ might be satisfied—in this case, extending the distribution to negative ω has only negligible consequences, if any.

Problem 3 (10 points) Assume that typographical errors committed by a typesetter are completely random. Suppose that a book of 1000 pages contain 1000 such errors.

(a) What is the probability that a page will contain no error?

(b) What is the probability that a page will contain three errors?

Problem 4 (10 points) A mirror is plated with gold by evaporation. The evaporation method works by heating a gold wire in a high vacuum. Gold atoms fly off in all directions and a portion of them sticks to the glass, or to other atoms already on the glass plate. Assume that each column of deposited atoms is independent of neighboring columns, and that the average deposition rate is d layers per second.

(a) What is the probability of m atoms deposited at a site after a time t? What fraction of the glass is not covered by any gold atoms?

(b) What is the variance in the thickness?

Problem 5 (15 points) By using the cluster expansion, find the following quantities in terms of cumulants and, when applicable, joint cumulants.

(a) $\left\langle x^5 \right\rangle$

(b) $\left\langle x^6 \right\rangle$

(c) $\left\langle x_1^3 x_2^2 \right\rangle$