Notes for Lecture 16

Landau theory

We consider the Ising model from the point of view of the mean field theory. We continue to discuss it from a slightly different, but related, point of view – the variational principle. This discussion will lead naturally to the Landau theory of phase transition.

16.1 Ising model (cont.)

We continue the study of the Ising model for the ferromagnetic case ($J > 0$), and develop important ways of thinking about it.

Let us start by checking some simple definitions.

$$M \equiv \sum_i \overline{\langle\sigma_i\rangle}\mu_B = N\overline{\langle\sigma_i\rangle}\mu_B \qquad \text{magnetization} \tag{16.1}$$

$$m \equiv x \equiv \frac{M}{N\mu_B} \qquad N = \text{the total number of spins} \tag{16.2}$$

Here, we just introduced the symbol m for **the magnetization in unit of μ_B per spin**. Or, it is **the fraction of the magnetic moments aligned parallel to the field**. In the previous lecture, we used x, the symbol for the unknown, in the last lecture. From this lecture note, we shall prefer m exclusively, when possible.

16.1.1 Gibbs free energy

We have, with rigor, used the symbol X for the partition function in the last lecture, since it really corresponds to the Gibbs partition function. That is, we have specified the H field as given, not M (cf. Sections 8.4.2,8.4.3).

With this understanding, we get, from Eq. 15.24, for the Gibbs free energy,

$$\begin{aligned} G_i(T,H) &= -k_BT\,\log\cosh(\beta H_i\mu_B) - k_BT\,\log 2 && \text{not quite right – see below} \quad (16.3)\\ G(T,H) &= -Nk_BT\,\log\cosh(\beta H_i\mu_B) - Nk_BT\,\log 2 && \text{not quite right – see below} \quad (16.4)\end{aligned}$$

Using Eq. 15.26 (with $x \to m$),

$$H_i\mu_B = Jzm + \mu_B H, \tag{16.5}$$

which leads to

$$G(T,H) = -Nk_BT\,\log\cosh\left(\beta Jzm + \beta\mu_B H\right) - Nk_BT\log 2 \quad \text{incorrect – see below} \tag{16.6}$$

In this expression, we must consider m as $m(T,H)$. All this seems fine, until we realize a problem! If one takes $-\left(\frac{\partial G}{\partial H}\right)_T$, one must get $M = mN\mu_B$, which must agree with Eq. 15.27. But, by doing this, we get

$$m\mu_B = \tanh(\beta Jzm + \beta\mu_B H)\left(Jz\frac{\partial m}{\partial H} + \mu_B\right) \quad \text{definitely incorrect!} \tag{16.7}$$

instead, which is *not* equal to Eq 15.27. What went wrong? The answer is the neglected term in Eq. 15.21. In the last lecture, we were focusing on the single spin dynamics only, and so we could ignore the last term of that equation, since it did not involve any dynamic variable (it was "just a constant"). However, from the point of view of the *total* Gibbs free energy, the last term, which is equal to

$$J\sum_{\langle i,j\rangle}\overline{\langle\sigma_i\rangle}\,\overline{\langle\sigma_j\rangle} = \frac{JNz}{2}m^2 \quad \text{division by 2, to compensate for double counting} \tag{16.8}$$

definitely cannot be ignored and must be included ($Nz/2$ is the total number of the nearest neighbor pairs). So, the Gibbs free energy must be written as

$$G(T,H) = -Nk_BT\,\log\cosh(\beta(Jzm + \mu_B H)) + \frac{JNz}{2}m^2 - Nk_BT\log 2 \tag{16.9}$$

Now, applying $mN\mu_B = -\left(\frac{\partial G}{\partial H}\right)_T$, one gets

$$\left(1 + \frac{\partial m}{\partial H}\frac{Jz}{\mu_B}\right)(m - \tanh(\beta Jzm + \beta\mu_B h)) = 0 \tag{16.10}$$

The quantity $\left(1+\frac{\partial m}{\partial H}\frac{Jz}{\mu_B}\right)$ is always positive for a ferromagnetic system (since the susceptibility $\frac{\partial m}{\partial H}$ is positive[1]) and thus we obtain the same correct result (Eq. 15.27) as we got in the last lecture.

$$\boxed{m = \tanh\left[\beta(Jzm+\mu_B H)\right] \qquad (15.27)}$$

16.1.2 Variational principle

As we discussed in the last lecture, the above equation has three solutions if $T < T_c$ where

$$T_c = \frac{Jz}{k_B} \qquad (16.11)$$

This temperature is (properly) the "critical temperature" as we shall see later on in this lecture. It is also called the Curie temperature in the case of the ferromagnetic model currently under consideration.

Now it is time to consider which of these three solutions are physical. The way to do it is to consider the Gibbs free energy of Eq. 16.9 as a function where m, now re-written as $\mathpzc{m}$, is the variational parameter that minimizes G, which we now rewrite as $\mathcal{G}$:

$$\mathcal{G}(\mathpzc{m},T,H) = -Nk_BT\log\cosh(\beta(Jz\mathpzc{m}+\mu_BH)) + \frac{JNz}{2}\mathpzc{m}^2 - Nk_BT\log 2. \qquad (16.12)$$

Having been written this way, this $\mathcal{G}(\mathpzc{m},H,T)$ is not yet the equilibrium Gibbs free energy. Rather, it is what appears in the exponent for X, as in[2]

$$X(T,H) = \sum_{\mathpzc{m}} e^{-\beta\mathcal{G}(\mathpzc{m},H,T)}. \qquad (16.13)$$

Note that we consider N as completely fixed for this problem (since spins are fixed in space), and thus treat N as a constant, throughout. So the current problem involves only two independent variables, which we take as H and T. As we noted in page 4 of LN 7, the equilibrium state is obtained by minimizing $\mathcal{G}$. **This fundamental**

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[1] It is, in particular, $+\infty$ below the transition.

[2] Here, $\mathcal{G}(\mathpzc{m},T,H) = \mathcal{E} - T\mathcal{S} - H\mathpzc{m}\mu_BN$, where $\mathcal{E} = \mathcal{E}(\mathpzc{m},H,T)$ and $\mathcal{S} = \mathcal{S}(\mathpzc{m},H,T)$. This formalism is not new at all. It follows from Eqs. 7.5 and 7.6, with $\mathcal{M} = \mathpzc{m}\mu_BN$ as the displacement (x in Eq. 7.6) and H as the force.

minimization principle for the free energy explains the variational principle whereby $\mathcal{G}(m, H, T)$ is minimized with respect to m.

Before we effect the minimization, let us express the free energy in terms of the following **dimensionless variables**:

$$t \equiv \frac{T - T_c}{T_c} = \frac{k_B T}{Jz} - 1, \tag{16.14}$$

$$h \equiv \frac{\mu_B H}{Jz}. \tag{16.15}$$

Then, we get, for the free energy per spin,

$$\begin{aligned} g(m, T, h) &\equiv \frac{\mathcal{G}(m, T, H)}{N} \\ &= k_B T \left[\frac{m^2}{2(1+t)} - \log\cosh\left(\frac{m+h}{1+t}\right) - \log 2 \right]. \end{aligned} \tag{16.16}$$

We are particularly interested in the behavior of the free energy near T_c, i.e. when m, $|t|$, and $|h|$ are all small. In particular, we will be interested in the response of the system in the leading order of $|t|$ and $|h|$: this means expanding the free energy up to linear order in these parameters. However, we note that m is not an input parameter at all. We do not know *a priori* how small it is, and so we must expand the free energy in enough orders of m so that our solution is physically sensible. For sensible results, we will find that keeping terms up to order m^4 is sufficient. Thus, using $\cosh(\delta) \approx \frac{1}{2}\delta^2 + \frac{1}{24}\delta^4$, and $\log(1 + \frac{1}{2}\delta^2 + \frac{1}{24}\delta^4) \approx \frac{1}{2}\delta^2 - \frac{1}{12}\delta^4$, we get $g(m, T, h)\beta \approx \frac{m^2}{2(1+t)} - \frac{1}{2}\left(\frac{m+h}{1+t}\right)^2 + \frac{1}{12}\left(\frac{m+h}{1+t}\right)^4$, which leads to the following result.

$$\boxed{g(m, T, h) \approx k_B T \left[\frac{t}{2} m^2 + \frac{1}{12} m^4 - mh - \log 2 \right] \qquad \text{near } T_c, \quad \text{small } H} \tag{16.17}$$

Here, for $|t|$ and $|h|$ terms we have kept only the linear order terms that involve the lowest order terms of m.

The above is an extremely important result[3]. For example, it shows that, if $T < T_c$ ($t < 0$) and $h = 0$, then $m = 0$ is a maximum of the free energy – thus, unphysical

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[3] In this note, this result has been derived from the mean field theory that we set up in the last lecture. An equivalent way to get this result is to use the so-called “Bragg-Williams approximation,” which is very much similar in spirit to the mean field approximation that we have employed here but is different enough in the method to give a somewhat different expression for g, whose expansion near $m, |h|, |t|$ is, nevertheless, identical with what we got here. This other method is discussed in standard textbooks, e.g. Plischke and Bergersen, or Huang.

– since the coefficient of the second order term is negative. Instead, two physical solutions that correspond to the minima of $\mathscr{g}$ emerge, since as $\mathscr{m}$ increases the next order term, $O(\mathscr{m}^4)$, forces the free energy to bounce back. **This proves that, of the three solutions that we obtained in the last lecture**

$$m = 0, \pm\bar{\sigma} \qquad \text{if } T < T_c \tag{15.29}$$

only $\pm\bar{\sigma}$ are physical solutions. Recall that, generally, $\pm\bar{\sigma}$ are the non-zero numerical solutions for $m = \tanh(\beta J z m)$, with $\bar{\sigma} > 0$ by definition. For $\mathscr{m} \approx 0$ and $h = 0$, they can be obtained perturbatively, from the two terms of the above equation, by setting $\frac{\partial \mathscr{g}}{\partial \mathscr{m}} = 0 \propto t\mathscr{m} + \frac{\mathscr{m}^3}{3} = 0$ as

$$m = \pm\sqrt{-3t} \approx \pm\sqrt{3}\left(1 - \frac{T}{T_c}\right)^{1/2} \qquad \text{for } T < T_c \text{ and } T \approx T_c. \tag{16.18}$$

On the other hand,

$$m = 0 \qquad \text{for } T > T_c \tag{16.19}$$

is the only physical state for equilibrium above the transition temperature.

Of course, these results could also have been obtained straight from the expansion of the solution for m (Eq. 15.27), which can be re-written as

$$\boxed{m = \tanh\left(\frac{m+h}{1+t}\right), \qquad \text{dimensionless form of Eq. 15.27.} \qquad (16.20)}$$

By setting $h = 0$ and using the expansion $\tanh\delta \approx \delta - \delta^3/3$, we get the same results for m near T_c (left for your exercise).

In any case, Eq. 16.18 means that $M \propto (T - T_c)^{1/2}$ near but below T_c. In the language of **the theory of critical phenomena**, this means that the so-called "β exponent"[4] is 1/2. It turns out that this mean field result is incorrect, in general. The correct value is 1/8 for $d = 2$, and ≈ 0.313 for $d = 3$, where d is the spatial dimension. However, $\beta = 1/2$ for $d \geq 4$.

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[4] The exponent β that defines $M \propto (T - T_c)^{\beta}$ has nothing to do with the temperature. It is unfortunate that we use β for two things, but, if necessary, please use the context to figure out which β we are talking about.

16.1.3 Curie-Weiss susceptibility

Let us consider the magnetic susceptibility per spin, in unit of μ_B per field strength (gauss, e.g.)

$$\chi(T) \equiv \frac{\partial m}{\partial H} \tag{16.21}$$

The susceptibility can be obtained from Eq. 16.20 by taking the derivative on H.

$$\chi = \frac{1}{\cosh^2\left(\frac{m+h}{1+t}\right)} \left[\frac{\chi}{(1+t)} + \frac{\mu_B}{Jz(1+t)}\right] \tag{16.22}$$

Assuming low field, and assuming that $m = 0$ (above T_c), we can approximate $\cosh\left(\frac{m+h}{t+1}\right) \approx 1$, and we get

$$\chi = \frac{\mu_B}{Jz}\cdot\frac{1}{t} = \frac{\mu_B}{k_B}\cdot\frac{1}{T-T_c} \qquad\qquad T > T_c \tag{16.23}$$

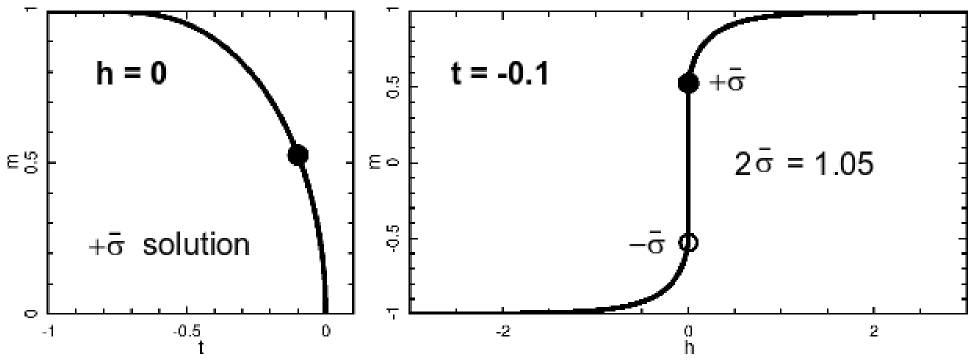
This is the **"Curie-Weiss law"** ($\chi \propto 1/(T-T_c)$) for a ferromagnet above the transition temperature. A special case, $J = 0$, corresponds to the $T_c = 0$ case, which gives the Curie's law $\chi \propto 1/T$, in agreement with what we learned in Section 8.4.2. For Curie's law to apply, one does not need a lattice of spins. Any number of non-interacting spins will give rise to Curie's law. For instance, a small amount of magnetic impurities give a large magnetic susceptibility at low temperature, showing Curie's law.

In any case, note that we just derived the magnetic susceptibility for any $T > T_c$. If one were to calculate the behavior for $T > T_c$ but only near T_c, then that would have been much easier to do. In that case, we can take Eq. 16.17, and minimize it for $\boldsymbol{m}$ to leading order in h, *and ignoring the* $\boldsymbol{m}^4$ *term since* $m = 0$ *without the* h *field*: $m = h/t = \mu_B H/(Jzt)$ is what we get. Thus, the susceptibility that one gets from this is exactly the same as the above result.

It turns out that the Curie-Weiss behavior is incorrect close to T_c, in general. Again, in the language of the theory of the critical phenomena, the magnetic susceptibility near T_c defines the so-called γ exponent ($\chi \propto (T - T_c)^{-\gamma}$), which is 1 within this mean field theory. The correct value is 7/4 in two dimensions, and ≈ 1.24 in three dimensions. It is 1 for dimensions four or higher. So, in general, the Curie-Weiss susceptibility is *not correct* near T_c. In fact, as we mentioned already, T_c itself is not correctly predicted by the mean field theory.

However, the Curie-Weiss law above T_c *is* important and useful. This is because the Curie-Weiss susceptibility *is* observed at high temperatures well above the transition temperature. This gives a method to estimate an important microscopic parameter such as J. This is the reason why we have derived this law for *any* $T > T_c$, in the above.

What is the susceptibility for $T < T_c$? To answer this question, it is most helpful to plot some curves, as shown below. The plot on the left shows m as a function of t, with zero field. So, this shows the **order parameter** that turns on below T_c, following the $\sqrt{t}$ behavior, near $t = 0$, as we showed in Eq. 16.18. The order parameter is ambiguous by sign, and here we choose to plot the $+\bar{\sigma}$ solution only. The plot on the right shows m as a function of h at $t = -0.1$, i.e., somewhat in the condensed phase. The two filled circles, one each in each plot, correspond to the same state.



Ising model, MF, magnetization as a funtion of t and h

Looking at the plot on the right, one can easily see that, within this theory[5], the magnetization has a finite jump, $2\bar{\sigma}$, at $h = 0$, for any $t < 0$. So, at precisely $h = 0$, the susceptibility is infinite for any $T < T_c$, due to this discontinuity of m at $h = 0$.

However, it is still meaningful (cf. footnote 5) to think about the magnetic susceptibility for small h, which does *not* cross zero. We can then proceed to calculate the susceptibility using Eq. 16.20, *assuming that m is a smooth function of h.* We have already wrote down the equation for χ using this assumption: Eq. 16.22. In fact, that equation is a general equation that will give the value of χ for any value of t and *non-zero h*. This will involve a numerical solution for m for a finite field, and it is left for your exercise.

Here, we focus on **the critical behavior**: small h and small t. If one takes the approximation that $\cosh^2\left(\frac{m+h}{1+t}\right) \approx 1$, then one gets a nonsensical result: a negative susceptibility! This is not surprising, actually. Ignoring m completely, we got the Curie-Weiss law, for $T > T_c$, above, from the same equation. This is clearly not

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[5] Note that the sudden finite step of m at $h = 0$ is *not* observed in experiments. Instead, hysteresis curves are obtained in experiments, as the field is cycled back and fro. The origin of this behavior is the magnetic domain formation.

appropriate for $T < T_c$. We *must* take into account the finite m value. Going to the next order, $\cosh^2\left(\frac{m+h}{1+t}\right) \approx \left(1+\frac{1}{2}\left(\frac{m}{1+t}\right)^2\right)^2 \approx 1+m^2$. Using this, and $m^2 \approx -3t$ (Eq. 16.18), we get $\left(\frac{t}{1+t} - 3t\right)\chi \approx \frac{\mu_B}{Jz(1+t)}$. Thus, we get

$$\chi = -\frac{1}{2t}\cdot\frac{\mu_B}{Jz} = \frac{\mu_B}{2k_B}\cdot\frac{1}{T_c - T} \qquad T < T_c \tag{16.24}$$

Comparing this result with the above result for $T > T_c$, Eq. 16.23, we see that both are governed by the same critical exponent, $\gamma = 1$, but they are not the same function of t.

Note that in order to calculate this critical behavior, it would have been sufficient to use Eq. 16.17 and evaluate m (in the presence of h) and then calculate $\frac{\partial m}{\partial h}$. If we do that, then we get

$$m^2 = -3t\left(1-\frac{h}{tm}\right) \qquad \text{small } t<0 \text{ and } h \tag{16.25}$$

$$\begin{aligned} m &= \pm\sqrt{-3t}\left(1-\frac{h}{tm}\right)^{1/2} \approx \pm\sqrt{-3t}\left(1-\frac{1}{2}\frac{h}{tm}\right) \\ &\approx \pm\sqrt{-3t} - \frac{1}{2}\frac{h}{t} \qquad m \approx \pm\sqrt{-3t} \end{aligned} \tag{16.26}$$

which agrees with the above, in that $\frac{\partial m}{\partial H} = -\frac{\mu_B}{2tJz}$.

16.1.4 Heat capacity

To calculate the heat capacity, we can use Eq. 16.9 or Eq. 16.16, while keeping in mind that $m = m(T, H)$. Here, we are interested in the near T_c behavior with $H = 0$. In this case, we can take Eq. 16.17 and Eq. 16.18 to express the free energy per spin as[6]

$$g(T) \approx -k_B T\left[\frac{3t^2}{4} + \log 2\right] \qquad \text{if } T < T_c, T \approx T_c \tag{16.27}$$

$$g(T) = -k_B T \log 2 \qquad \text{if } T > T_c \tag{16.28}$$

Note that for the last equation, we have extended the region of validity for T to an arbitrarily high temperature, not just near T_c, by using Eq. 16.16. Thus, the entropy

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[6] Just to remind you, we have been using such as $\mathcal{m}$, $\mathcal{g}$, $\mathcal{G}$, $\mathcal{S}$, $\mathcal{E}$, etc., to represent random variables, and m, g, G, S, E, etc., to represent corresponding equilibrium thermodynamic variables, throughout this course. This practice is more rigorous than any common textbooks, and is designed to give a firmer grasp of the underlying physics. However, in some case (Landau free energy), we might break this rigorous convention.

per spin is given by

$$\frac{S}{N} = -\frac{\partial g}{\partial T} \begin{cases} \approx \frac{3k_B}{2} \cdot \frac{T-T_c}{T_c} + k_B \log 2 & \text{if } T < T_c, T \approx T_c \\ = k_B \log 2 & \text{if } T > T_c \end{cases} \tag{16.29}$$

As expected, we get $k_B \log 2$ for the high temperature regime. The heat capacity per spin is

$$\frac{C}{N} = \frac{T}{N}\frac{\partial S}{\partial T} \begin{cases} \approx \frac{3}{2}k_B & \text{if } T < T_c,\ T \approx T_c \\ = 0 & \text{if } T > T_c \end{cases} \tag{16.30}$$

Therefore, there is a jump in the heat capacity. Note on the other hand that the entropy is continuous. Is is said that we have a *second order* phase transition: the first derivative of the thermodynamic potential G is continuous, while the second derivative of G is discontinuous. If the entropy (or other first derivative) is discontinuous, then it is said that we have a *first order* phase transition. We have seen this in the BEC case, as the latent heat L, given by $L = T_c \Delta S$, was finite in that case (Section 15.1.4).

Once again, the mean field theory is pointing out some interesting physics, the discontinuity in the heat capacity, but if one looks at the matter carefully it will be found that the mean field theory did not get the quantitative answer correct. Indeed, the singularity of the heat capacity is more severe in experiment or in a more sophisticated calculation. In the critical phenomena language, one defines the "α exponent" as $C \propto |t|^{-\alpha}$ near T_c. So, within the mean field theory of the Ising model, we have $\alpha = 0$. This happens to be sort of[7] correct for the Ising model in two dimensions. However, the correct value of α for the Ising model in three dimensions is ≈ 0.11.

16.2 Spontaneous Symmetry Breaking

From the above considerations, we have seen that the system has available two ground states $m = \pm\bar{\sigma}$ for $T < T_c$. This situation is summarized in the diagram below. With $h = 0$ and $T < T_c$, the two minima define two possible equilibrium states. Note that any small field h will skew the potential towards one of the two minima.

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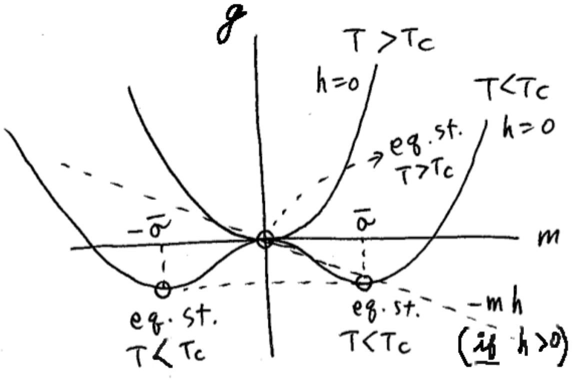
[7] The correct solution agrees in that there is no power singularity, but disagrees in that the correct solution diverges at T_c logarithmically.



Let us ask some basic questions. How are these two states (with $m = \pm\bar{\sigma}$) related to each other? If a system somehow forms in one state, can it evolve to the other state? The answer to the first question is that these two states are related to each other by the symmetry of the underlying Hamiltonian: they are degenerate eigenstates, connected by the spin inversion. The answer to the second question is that they *cannot* spontaneously transition to the other state.

This means that one of the two states will be a macroscopic state that will be realized.[8] Now, one can note that the physical state is then in a *lower* symmetry state than the Hamiltonian. The Hamiltonian (Eq. 15.13) is invariant under the reversal of all spin directions ($\sigma_i \to -\sigma_i$). Clearly, a state in which all spins are pointing along one direction is *not* symmetric upon the reversal of all spin directions.

Then, we conclude that the condensed state of a ferromagnet is a spontaneously formed state that has a lower symmetry than the Hamiltonian. This is the very important concept of **the spontaneous symmetry breaking**.

Here, "spontaneous" means without any external help, i.e., we must ensure that there is zero external field. Namely, if you just cool down a ferromagnet it will be magnetized along a certain direction according to our theory. For an Ising magnet that we have been considering, it is not possible to tell which of the two directions that the spins will be aligned to. But, it is certain that the system will choose one direction. **Had we considered a Heisenberg ferromagnet, then the choice of the direction would be continuous – and thus the magnet has infinite choices for its direction! However, the system will choose one of those states.**

The concept of the spontaneous symmetry breaking is very fundamental. If somehow you can imagine that we were ant-like creatures living in space which turns out

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[8] Of course, a real material can form domains. We are not really considering domain formations here, just yet. But, as you will see, soon, we will have to take them into consideration!

to be a giant Ising magnet of some sort, then we might feel that our world is somehow not symmetric. Such a magnetization that is *part* of the space that we live in will break the time reversal symmetry and the inversion/parity symmetry. We might feel that this is very strange. While we would assert that physical laws must be symmetric, we would have to formulate physical laws in the "magnet" in accordance with these broken symmetries, and so it will look like those laws are actually not symmetric. This is the analogy to what we know is happening in the weak interaction case, where the parity and the time reversal (and the charge conjugation) symmetries are violated. In essence, one can consider the electro-weak theory as the theory that emerges in a broken-symmetry phase of vacuum.

We have yet to show that the broken symmetry state is stable. That is, one state does not evolve to another state spontaneously. But before we do the math, let us consider physics of magnets a bit more.

We have been considering a particular model of a ferromagnet. How about a real ferromagnet? A real magnet is likely a Heisenberg magnet than an Ising magnet, and so you may feel that infinite choices of continuous values for the direction mean that a magnet can "slowly transition" so that its magnetization direction points slightly differently. If such a process was possible, then the magnetization direction will slowly evolve to some completely different direction. But, we *know* that this does *not* happen spontaneously in a real magnet. Namely, when one has in hand a magnet, then that magnet will stay in a magnetized state for a long time, unless something dramatic is done to it. By something dramatic, I mean something like a temperature cycling *up through* the Curie temperature and then down, or an application of a large magnetic field. This type of dramatic disturbance can change the magnetization direction. However, we expect a magnet to stay magnetized in one direction, if we keep it at one temperature without any disturbance. Otherwise, a magnet would be completely useless, e.g., as a compass!

Why is it then that a spontaneously broken symmetry is stable? It is in fact due to the same reason why the Poincaré period is very large and why an entropy decreasing process practically never occurs. Suppose that a spin has a probability p to change its direction to a specific direction. In an Ising magnet, this number will be given by $\exp(-\beta O(J))$, since the spin has to flip in an Ising magnet. In a Heisenberg magnet, there would not be a large activation energy required, since a spin can cant just a little to change its direction. However, one can see that the direction of canting can be chosen to be any direction. So, the probability of p for one certain direction will be very small. No matter what the value of p is however, what is important is that it is *not* 1. Now, the probability for all the spins to align along the same direction is given by p^N, where N is the number of spins. For any number $0 \le p < 1$, $p^N \to 0$ for $N \to \infty$, so there is zero chance that a spontaneously broken symmetry state will transition to another state.

This does not mean that system does not remember the symmetry it broke. If the broken symmetry is continuous, as in the Heisenberg magnet case, the following happens. While it is impossible to make all spins suddenly point to another direction, a small local disturbance of canted spins can and do happen. This type of disturbance can propagate as a wave. If the canting of spins is very small on local scales, then the restoring force required vanishes, assuming that spin-spin interaction is not long-ranged. This means that $\omega(\vec{k}) \to 0$ for $\vec{k} \to 0$. This wave is called "magnon" or "spin wave." The magnon is an example of a Goldstone boson. We already mentioned the Goldstone theorem that states that a broken continuous symmetry implies the emergence of a boson mode, whose energy goes to zero at long wave length. You may remember that we already saw an example of a Goldstone boson – the sound wave, i.e. the acoustic phonon or the Debye phonon, in a solid (LN 11, page 5).

16.3 Order parameter and the Landau theory

The quantity m (or M) is an example of an **order parameter**, a parameter that turns on below the transition temperature and marks the *lower symmetry* in the condensed state. So, it is a parameter that signals a spontaneous symmetry breaking. For an Heisenberg magnet, the order parameter would be vectorial, $\vec{m}$. For a superfluid or a superconductor, the order parameter is a wave function normalized to the condensate density, $\psi(x)$ (cf. Lecture 15). For a gas-liquid transition, the order parameter is the density. For a ferroelectric transition, the order parameter is $\vec{P}$, the polarization.

Landau proposed a simple phenomenological theory of phase transition, which has more than proved its value over time. The Landau theory starts out as a mean field theory of the kind that we have been discussing now. Its focus is the symmetry and the behaviors near the transition. **It assumes that the Landau free energy is analytic near the transition point**, just as we found above (as well as in the case of van der Waals theory of gas-liquid transition – which should not surprise you at this point now that we have established the similarity between the two problems, using the lattice gas picture). **The Landau free energy that we are considering is in fact the free energy as a random variable, rather than its minimized value, which would correspond to the equilibrium free energy.** For different problems, it may correspond to different free energy, such as the Gibbs free energy or the grand potential. Regardless, the Landau formalism relies on the fact that such a random variable corresponding to a thermodynamic potential must be minimized in equilibrium (LN 7, page 4). Thus, when we do minimize the Landau free energy with respect to all possible values of the order parameter, then we will be obtaining the equilibrium free energy, and all other related quantities, as we have been demonstrating in this lecture.

We must note that while the Landau free energy was an analytic function of m, h and t (see Eq. 16.17), all results that we obtained on m, χ, S, C etc. are *not* analytic at the critical point $t = 0$. Why did this happen? It is because something abrupt happens as the new condensed phase sets in. It is instructive to examine the diagram given in page 10. Notice how smooth and nice each curve for $\mathcal{g}$ is (two are shown: one for $T > T_c$ and the other for $T < T_c$). Now, note how the *equilibrium* Gibbs free energy determined by minimizing $\mathcal{g}$ (indicated by open circles) "breaks off" of the origin as T departs from T_c and decreases. **This explains why the thermodynamic functions become non-analytic around a phase transition.**

Now, we state the Landau theory in its simplest form. In fact, we have already covered this form above! One difference is that from now on, when we write down the Landau free energy, we will be a bit sloppy in our notation, as is standard in the field, and will not distinguish explicitly a random variable and an equilibrium value (see footnote 6). For a symmetric system ($h = 0$) with a real scalar order parameter m, the Landau free energy is defined as

$$G(m,T,h=0) \;=\; a(T) + \frac{1}{2}b(T)m^2 + \frac{1}{4}c(T)m^4 + \ldots \tag{16.31}$$

in accordance with the spontaneous symmetry breaking. Keep in mind that $T \approx T_c$ and m is small, by assumption. For a second order phase transition, we require that

$$b(T) \;=\; b_0\,(T - T_c), \qquad\qquad b_0 > 0, \tag{16.32}$$

$$c(T) \;>\; 0. \tag{16.33}$$

In the more general case when the (generalized) field h is non-zero, we have, within a linear approximation for coupling of h and m,

$$\boxed{G(m,T,h) \;=\; -hm + a(T) + \frac{b_0}{2}(T - T_c)m^2 + \frac{1}{4}c(T)m^4 + \ldots \qquad (16.34)}$$

In this sloppy standard notation, it is now critically important that a reader knows clearly that m and G are random variables, and that only when G is minimized, $m = m(T,h)$ becomes the equilibrium magnetization and $G(T,h) = G(m(T,h),T,h)$ becomes the equilibrium free energy. The two independent variables are taken as T and h. Then, m is *not* an independent thermodynamic variable[9], but *just a variational parameter.*

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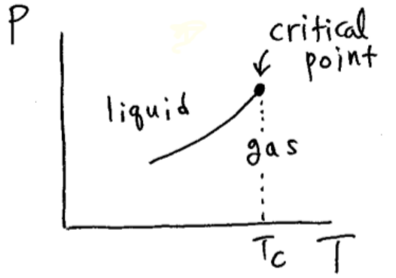
[9] So, it might be a good practice to write something like $G(m;T,h)$. However, in keeping with the sloppy standard notation, we will not do this here. In the case of the Landau theory, this sloppiness is understandable. The order parameter m is always *just a variational parameter.*

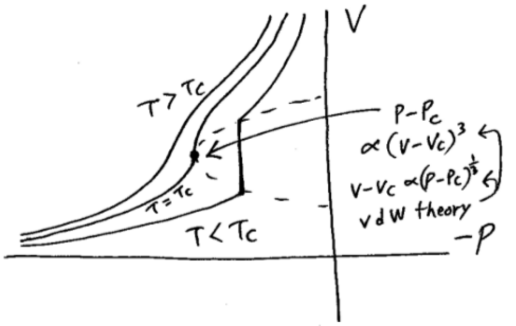
As mentioned above, G can be any possible free-energy or potential[10], since in this general theory we do not specify what h and m actually are. m can be any real scalar order parameter, not just the magnetization, and h is its conjugate field, both made dimensionless[11]. And of course, the theory can be generalized to vectorial or complex order parameters.

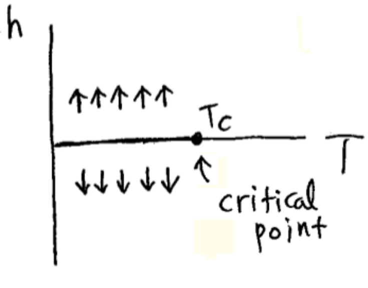
By minimizing G, we can calculate various thermodynamic functions such as m, χ, S, C, etc. near T_c, as function of h, a, b_0 and c. This is left for your work. In particular, one can identify what a, b_0 and c are in the case of the Ising model, and reproduced all results that we have obtained in this lecture near T_c.

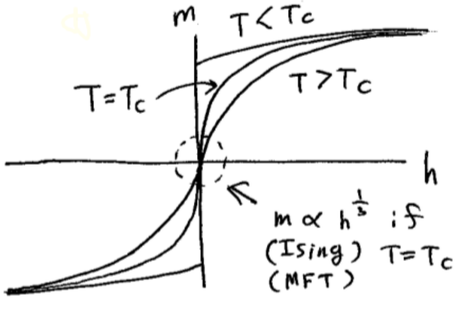
16.4 Phase diagrams

As a summary, the following diagrams might be helpful.









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[10] The textbook uses the symbol ψ for G.

[11] Note that if h and m are dimensionless, then G is dimensionless. This is not a problem. As usually is the case in the Landau theory or the RG theory, the energy can be made dimensionless if divided by k_BT. Keeping the meaning of m and h the same as in the rest of this lecture note, we can interpret the above Landau free energy G as $\frac{g}{k_BT}$, where g is given by Eq. 16.17. This is not a unique way to set up the problem, though. One *can* take G as the true total Gibbs free energy, for example. Then, we must interpret h as Nk_BT_ch, where the second h is the dimensionless h used in the other parts of the lecture. In such a case, the h that appears in the Landau free energy will not be dimensionless.

These diagrams compare the well-known gas-liquid phase transition (the upper two diagrams) with the magnetic transition that we have been discussing here using the Ising model. Note the similarities and some differences. One of the main differences is the symmetry in the latter problem with respect to flipping spins, which causes the phase coexistence line to be flat in the h-T diagram and the isotherms to have a fixed point at origin in the m-h diagram.

In the diagram, it is indicated that the equation of state at $T = T_c$ has the same behavior in the two systems: $h \propto m^3$ and $P_r - 1 \propto (v_r - 1)^3$ (Homework 5). This is not surprising at all, since we have already shown that the gas-liquid transition can be described by the Ising model, and both the van der Waals theory and the Landau theory are mean field theories. In the critical phenomena language, this means that the "δ exponent" is 3 for this class of models. The correct exponent for the Ising model is 15 for two dimensions and $\approx$ 4.789 for three dimensions.

Lastly, note that this diagram does identify the Curie temperature as the critical temperature. A second order phase transition occurs at that point. On the other hand, if $T < T_c$ and h is swept, we get a first order transition, where the "volume" ($=$ m) changes abruptly. Note, however, that in this case, there is *no* latent heat, as can be seen from the symmetry and also from that fact that the phase coexistence line is flat (cf. Clausius-Clapeyron equation).